AS2001/AS2101 Exoplanetary Science

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These notes contain all the important information presented within the lectures (the slides of which will also be made available).

Section 1: What is an exoplanet? Where do they form from? How do we spot them?

What is an exoplanet?

It is not:

- **A star**: so, no Hydrogen burning
  $M < 0.08\ M_{sun}$ (80 M_{jup})
- **A brown dwarf**: so, no Deuterium burning
  $M < 0.01\ M_{sun}$ (13 M_{jup})

Two options:

1. Object that forms in a circumstellar disc

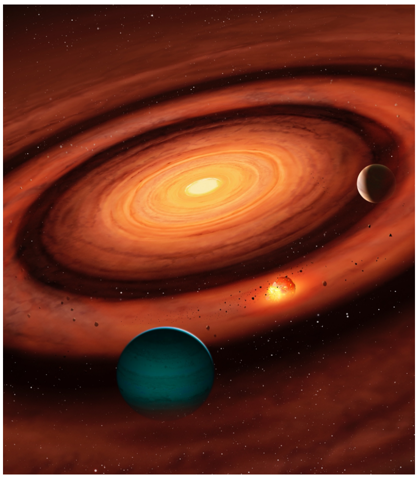
- by-product of star formation
- still needs to be a low-mass object (not another star)

The problem with this definition is: what about free-floating planet-mass objects? (Rogue planets)

2. A low-mass object that forms in a disc through planetesimal accumulation.

This means the metallicity of planets is higher than that of the host star.
This has other problems – sometimes the metallicity of stars and planets within a system are not as you might expect.

Should all stars have planets?



1. The nebular hypothesis (NH):

Kant (1755) and Laplace (1796) asked the question of "what happens if a star forms for a rotating cloud of gas?" Their answer was:
A collapsing protosolar nebula flattens as it rotates and planets condense within in disc.

This method predicts that all stars should be born with planets.

There were some objections to this theory. An example is James Clerk Maxwell in the 19th century who countered that:

- the Sun rotates slowly but has >99% of the Solar System mass.
- The planets contain 0.1% of the mass but >99% of the Solar System's angular momentum.

How then could the mass and angular momentum be separated? He proposed that the issue with the nebular hypothesis was that parts of the star's disc rotate with different velocities, so Keplerian shear within the disc would prevent planet formation (or so he thought… and everyone believed him!)

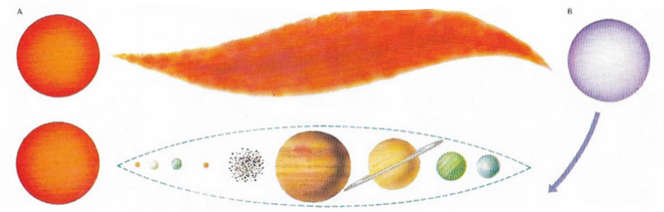
An alternative to this theory is:

2. The tidal model:

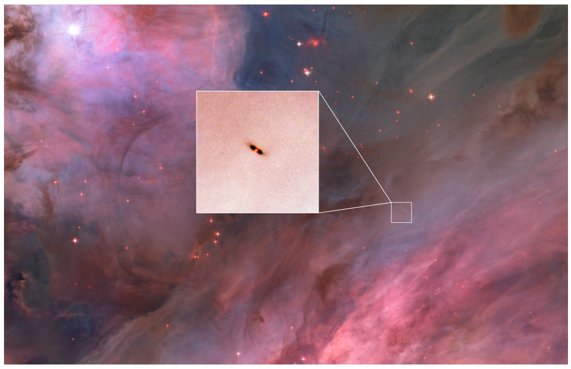
Jeans (1917) proposed that maybe a closer encounter of the Sun with another star created a tidal bulge from the Sun, that settled down into the planetary system.

The problem with this is that it relies on stellar encounters, and these are rare!

This method predicts that planetary systems should be rare.



Who was right? Data from the Hubble Space Telescope in the 1990s showed protoplanetary discs silhouetted against the Orion Nebula. Kant and Laplace were right, we see protoplanetary discs, and with the new generations of telescopes we can get even more detailed images!



Credit: NASA, ESA (HST Orion Treasury Project Team)

Star formation in galaxies:

- Planets are by-products of star formation.
- Star formation occurs in spiral arms.
- Stars form in molecular clouds.

Molecular clouds typically have the following properties:

- Densities: 10^{-19} to 10^{-16} kg m^{-3}
- Sizes: few 10's of parsecs : few 10^{17} m
- Mass: 1000's to 10^5 M_{sun}

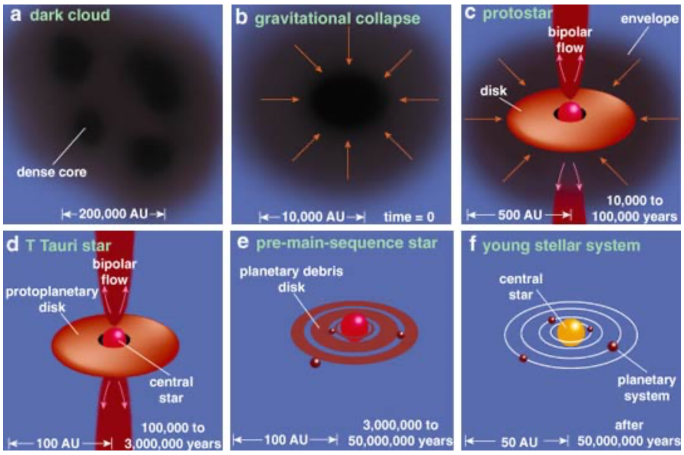
A molecular cloud this size must be reduced down eventually to a star like the Sun, which has the following properties:

- Densities: 1000 kg m^{-3}
- Size: 7×10^8 m

We can now resolve the structure of these discs with telescopes like ALMA and in doing so it is clear that there is a wide variety in disc structure for example:

- gaps,
- rings,
- hollowed out centres,
- extended discs,
- compact discs…

<u>Stages of star formation:</u>

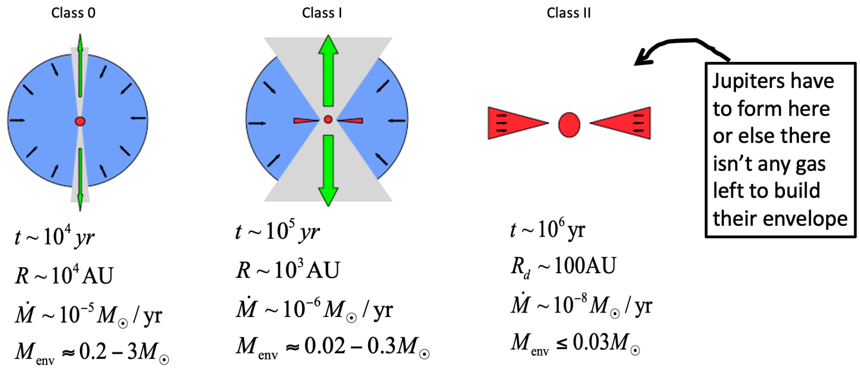


Prior to formation, there are molecular cloud within the Galaxy and experiencing the galactic dynamics. Eventually this cloud experiences gravitational collapse to form a protostar. Protostars can be classified into 4 classes:

- Class 0 -- youngest
- Class 1
- Class 2 (aka TTauri)
- Class 3 -- oldest

<u>Class 0:</u> The cloud is collapsing because $E_{internal} + E_{grav\ binding} < 0$ (see later). Gas and dust are infalling onto the low-mass protostar core, and we can detect in far IR and mm (~100mm to mm) but not optically.
<u>Class 1 (Class I):</u> The dense centre that is now about the size of a stellar mass object. The accreting (infalling) material collides with itself as it falls, and so radiates energy. There is now formation of a protostellar disc, and the object is visible in IR (~10mm).
<u>Class 2 (Class II) / TTauri:</u> The star is still contracting and is surrounded by a disc, and it may have outflows. Gas is accreting onto the star i.e. there is disc accretion onto a pre-main sequence (PMS) star. The object is optically visible with a disc (~0.6mm).
<u>Class 3 (Class III):</u> The gas is now gone but the dust remains. We now can no longer see the disc. Planets are forming and the object is now a pre-main sequence (PMS) star of final mass. The star is optically visible but with no disc.

We aren't entirely sure how the disc disperses; it may be because of the stellar wind (but there is a lot we don't know!).



In this course we will talk about planets around solar mass stars because for high mass stars it can be harder to observe the stars. Planets form within **protoplanetary discs** around young stars. Understanding the formation and evolution of these discs is central to a theoretical understanding of extrasolar planets.

Initiating core collapse:

To collapse, the total gravitational energy pulling the cloud in **must be greater than** the kinetic (i.e. thermal) energy pushing out. To form a star, a cloud with radius R and temperature T must have a mass M great enough to satisfy the condition for collapse. This critical mass M_J is named the ***Jeans mass*** (after Sir James Jeans, who figured it out first).

$$|E_{\rm grav}| \geq E_{\rm support} \approx E_{\rm thermal}$$

$$|E_{\rm grav}| = \frac{3}{5}\frac{GM^2}{R}$$

$$E_{\rm thermal} = \frac{M}{\mu m_{\rm H}} \times \frac{3}{2}kT = \frac{3}{2}M \times c_{\rm s}^2$$

μ = Mean molecular

m_H=Mass of H atom

c_s=sound speed

Disc formation:

Molecular clouds rotate because they orbit the galaxy (in ~3x10^8 yr). Because of this, they contain some angular momentum J. This is unlikely to be zero because there are lots of bulk motions. Angular momentum is conserved during collapse.

Angular momentum per unit mass (*"specific angular momentum"*) can be calculated by:

$$J = \vec{r} \times \vec{v}$$

During collapse the radius decreases and thus **v_{rot} increases**

A "centrifugal barrier" halts the collapse: at some point, v increases sufficiently to the halt collapse.

$$v = \frac{J}{r} \Rightarrow \frac{v^2}{r} \approx \frac{Gm}{r^2} \Rightarrow r_{cent} = \frac{J^2}{Gm}$$

The collapse is halted perpendicular to J, which leads to disc formation. Later there will be planet formation. So how much do molecular clouds shrink when they form a star + disc?

Worked example:

A cloud with mass M = 1 M_{sun} , radius R = 0.1 pc is rotating with the galactic rotation period P = 250 Myr (i.e. it's rotation matches the galaxy's).
What is the orbital velocity of the material within the cloud?

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{2.5 \times 10^8 \times 365.25 \times 86400} = 7.96 \times 10^{-11} \text{ radian s}^{-1}$$

What is its angular momentum per unit mass at outer edge?

$$J = \omega r^2 = 7.96 \times 10^{-11} \times (3.086 \times 10^{15})^2 = 7.58 \times 10^{15}\ \mathrm{m^2 s^{-1}}$$

Now we can calculate the radius that the cloud will shrink to (remember the centrifugal barrier which halts the collapse?)

$$r = \frac{J^2}{GM} = \frac{(7.58 \times 10^{15})^2}{6.674 \times 10^{-11} \times 1.989 \times 10^{30}} = 4.32 \times 10^{11}\ \mathrm{m}$$

In other words, r ~ 2.9 AU, this is about the distance from the Sun to the edge of the asteroid belt. A small amount of mass must move *out*, carrying away angular momentum, to allow most of the mass to move *in* and form the star.

Finding planets:

Why are planets so hard to find, since planets are the by-product of star production, they should be everywhere…

At visible wavelengths, planets shine by reflected starlight.
Inverse square law tells us that planet at distance *a* receives the flux:

$$F_p(\lambda) = F_*(\lambda)\frac{R_*^2}{a^2}$$

An observer at distance, *d*, with *d* >> *a* receives flux from the star of:

$$f_*(\lambda) = F_*(\lambda)\frac{R_*^2}{d^2}$$

The same observer also receives flux from the planet of:

$$f_p(\lambda) = p(\lambda)F_p(\lambda)\frac{R_p^2}{d^2}$$

p < 1 is the *geometric albedo* of the planet and is the fraction of the flux reaching the planet that is reflected from the surface.

The ratio of fluxes received from planet and star is:

$$\epsilon \equiv \frac{f_p(\lambda)}{f_*(\lambda)} = p(\lambda)\frac{R_p^2}{a^2}$$

As some examples: Venus, Earth and Jupiter have:

- visual geometric albedos *p* = 0.67, 0.37, 0.52;
- radii R_p = 6052, 6371, 69911 km;
- orbital distances *a* = 1.082×10^8, 1.496×10^8, 8.157×10^8 km.

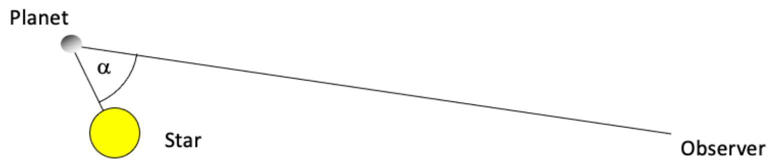
Venus has

$$\epsilon = 0.67\left(\frac{6052}{1.082 \times 10^8}\right)^2 = 2.1 \times 10^{-9}$$

Q0. Do the other two yourself below. Which planet is brightest to an interstellar observer?

But this flux ratio just calculated assumes we are looking directly at the planet's dayside, which would put the planet directly behind the star. We need to put in some term to tell us about the phase of the planet… The **phase function** corrects for partial illumination as a function of the phase angle subtended at the planet by the star and the observer.

$$\Phi(\alpha) = \frac{\sin(\alpha) + (\pi - \alpha)\cos(\alpha)}{\pi}$$



Ultimately this is an approximation and not perfect because it can vary with planet/moon composition, but it is still useful.
The phase function doesn't account for a planetary atmosphere. Planets like Venus with an atmosphere can still be seen even when alpha = 180 i.e. on their night-side. This means their phase function would not equal 0 at this angle (whilst it would for a planet like Mercury!).

Looking in infrared:

Planets are warm due to the starlight they absorb, but they are still cooler than the star. They re-emit that starlight as thermal/IR light and this planetary thermal emission can >> optical reflected light.
Of course, the star also has IR emission… Even if we can resolve the star from the planet, there isn't much contrast so we might want to know what contrast would we see in the IR?
The thermal flux received from an object with temperature *T* at frequency n = *c* / l is described by

$$\text{flux} = \text{surface brightness} \times \text{solid angle}: \qquad f_\nu = B_\nu(T)\,\frac{\pi R^2}{d^2}$$

(Solid angle = surface area of object/distance from star)

We can obtain the brightness from the Planck function:

$$B_\nu(T) = \frac{2h\nu^3/c^2}{\exp(h\nu/kT) - 1}$$

At thermal IR frequencies, $h\nu << kT$ (i.e the photon energy is much less than the mean kinetic energy of the particles within the object), so we can use the *Rayleigh-Jeans approximation*:

$$\exp(h\nu/kT) \simeq 1 + (h\nu/kT)$$

Hence,

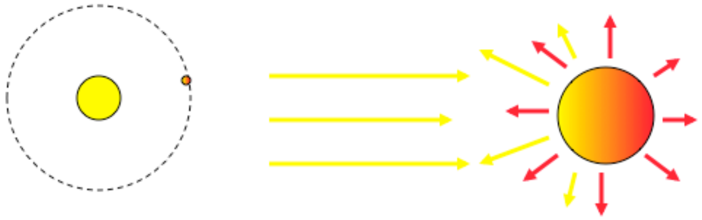
$$B_\nu(T) \approx \frac{2\,k\,T\,\nu^2}{c^2} = \frac{2\,k\,T}{\lambda^2}\,.$$

So, the brightness is proportional to the temperature and 1/wavelength^2.

Equilibrium Temperature:

The equilibrium temperature of a planet is the temperature that a planet would be at if it was in radiative equilibrium. Usually, we assume that the planet radiates as a black body that is only heated by its host star. Aside: Can you think of any reason why these assumptions might not be perfect?
We wish to get an equation that would allow us to calculate the equilibrium temperature of planets. Start with:

Power input (from the star) **= Power output** (from the planet's reflected starlight + thermal radiation)



Remember that power is luminosity, which we can calculate from the flux received from the star. And remember the equation $P = \epsilon\, \sigma\, A\, T_*^4$?

The power of the star is given by $L_* = \epsilon\, \sigma\, T_*^4\, Area = \sigma T_*^4 (4\pi R_*^2)$ (where we assume the star is a perfect blackbody so $\epsilon = 1$) and the stellar flux that reaches the planet then is $F_{in} = \frac{L_*}{4\pi a^2}$ where *a* is the distance between the star and planet.

The power outputted from the planet is $L_{reflected} + L_{thermal}$.

The power reaching the planet is $F_{in}(\pi r^2)$, where *r* is the radius of the planet, but not all of this is reflected. The reflected power is given by:

$$L_{reflected} = F_{in}(\pi r^2)A$$

Where *A* is the ***bond albedo*** and tells us about the fraction of the power that gets reradiated into space. It has a value between 0 and 1. Something with a bond albedo of 1 reflects everything back into space, whilst an object with a bond albedo of 0 reflects nothing.

The planet in this scenario is in thermal equilibrium. So, the power in = power out. If

$$L_{reflected} = F_{in}(\pi r^2)A$$

gives the reflected power, then whatever remains must be radiated away i.e. the radiated power:

$$L_{radiated} = L_{in} - L_{reflected} = F_{in}(\pi r^2) - F_{in}(\pi r^2)A = F_{in}(\pi r^2)(1 - A).$$

We can also calculate this thermally radiated power from the equation $P = \epsilon\, \sigma\, A\, T_*^4$, again assuming a perfect blackbody.

$$L_{radiated} = \sigma(4\pi r^2)T_p^4$$

Now we can calculate the equilibrium temperature of the planet, T_p, by setting these two expressions for$L_{radiated}$ to be equal:

$$F_{in}(\pi r^2)(1-A) = \sigma(4\pi r^2)T_p^4$$

And using the expression for F_{in} at the top of this derivation:

$$\frac{L_*}{4\pi a^2}(\pi r^2)(1-A) = \frac{\sigma T_*^4(4\pi R_*^2)}{4\pi a^2}(\pi r^2)(1-A) = \sigma(4\pi r^2)T_p^4$$
$$\frac{T_*^4(R_*^2)}{4a^2}(1-A) = T_p^4$$
$$T_p = T_*\left(\frac{R_*}{2a}\right)^{1/2}(1-A)^{1/4}$$

This is the temperature that the planet must have if it radiates and reflects away the same amount of power as it receives from the host star.

Note that T_{eq} **is independent of the planet size** $\boldsymbol{r}$**!**

Worked example 2:

a) Estimate Jupiter's equilibrium temperature, given:

- T_* = 5780 K
- R_* = 6.96 x 10^5 km
- a = 8.157 x 10^8 km
- A_J = 0.343

$$T_J = T_*\left(\frac{R_*}{2a}\right)^{1/2}(1-A)^{1/4}$$
$$= 5780\sqrt{\frac{6.96\times10^5}{2\times8.157\times10^8}}(1-.343)^{1/4}$$
$$= 107.5K.$$

b) Given Jupiter's equilibrium temperature T_J = 107 K, calculate the ratio of fluxes received from Jupiter and the Sun, as seen at interstellar distance d.

$$B_\nu(T) \approx 2\,k\,T/\lambda^2$$
$$\text{flux} = \text{surface brightness} \times \text{solid angle}: \quad f_\nu = B_\nu(T)\,\frac{\pi\,R^2}{d^2}$$

$$\frac{f_J}{f_\odot} = \frac{2kT_J/\lambda^2}{2kT_\odot/\lambda^2}\frac{\pi R_J^2/d^2}{\pi R_\odot^2/d^2} = \frac{T_J}{T_\odot}\frac{R_J^2}{R_\odot^2}$$
$$= \frac{107\mathrm{K}}{5780\mathrm{K}}\left[\frac{6.99\times10^4\mathrm{km}}{6.96\times10^5\mathrm{km}}\right]^2 = 1.86\times10^{-4}$$

In visible light, the ratio is 10^{-9} so this is far more favourable!

Q1. An exoplanet has an equilibrium temperature of 180K and orbits a K type star with radius R=0.8Rsun and effective temperature 5000K. The planet is observed to orbit at a distance of 0.9AU. Calculate the albedo of the planet.

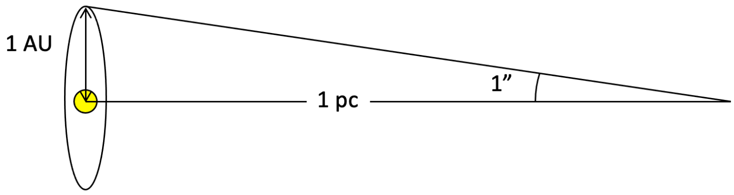
Q2: What are the equilibrium temperatures for the following bodies in the solar system? Given $T_* = 5780$ K and $R_* = 6.96 \times 10^5$ km.

Body	**Bond albedo, A**	**Distance from Sun, a**
Mercury	0.088	0.3871 AU
Venus	0.76	0.7233 AU
Earth	0.306	1 AU
Moon	0.11	1 AU
Mars	0.25	1.5273 AU
Enceladus	0.81	9.5AU
Pluto	0.41	39.5294 AU
Eris	0.99	67.6681 AU

Which of these celestial bodies reflect a lot of sunlight back into space, and which must reradiate a lot of sunlight?

A reminder on angular separation:
The definition of a parsec is *the distance at which the mean radius of the Earth's orbit subtends an angle of one second of arc*.

In other words: one AU subtends a parallax of 1 arcsecond at a distance of one parsec.



Q3. To an observer 10 pc away, what is the angular separation of the Earth and the Sun?

From observational techniques you have met the equation:

$$\theta = 1.22\frac{\lambda}{D.}$$

The atmospheric seeing from a really good site on a really good night is about 0.5” and the diffraction limit of a telescope of diameter *D* observing at wavelength λ is then given by the above equation.

Worked example 3:

Ignoring atmospheric seeing, how big a telescope is needed to resolve Earth from the Sun at a distance 10 pc and wavelength 500 nm?

$$\theta = 1.22\frac{\lambda}{D}$$

$$\theta = 0.1\text{”} = 0.1\frac{\pi}{180 \times 3600}$$

$$\Rightarrow D = 1.22\frac{\lambda}{\theta} = 1.22\frac{500 \times 10^{-9} \times 180 \times 3600}{0.1\pi} = 1.25\text{m}$$

Summary:
The planet/star flux ratio in reflected light at phase angle α is:

$$\epsilon(\lambda, \alpha) \equiv \frac{f_p(\lambda)}{f_*(\lambda)} = \Phi(\alpha)p(\lambda)\frac{R_p^2}{a^2}$$

Planet equilibrium temperature depends on stellar temperature, bond albedo, stellar radius and distance:

$$T_p = T_*\left(\frac{R_*}{2a}\right)^{1/2}(1-A)^{1/4}.$$

Thermal infrared planet/star flux ratio depends on the temperatures and areas of both bodies:

$$\frac{f_p}{f_*} = \frac{T_p R_p^2}{T_* R_*^2}.$$

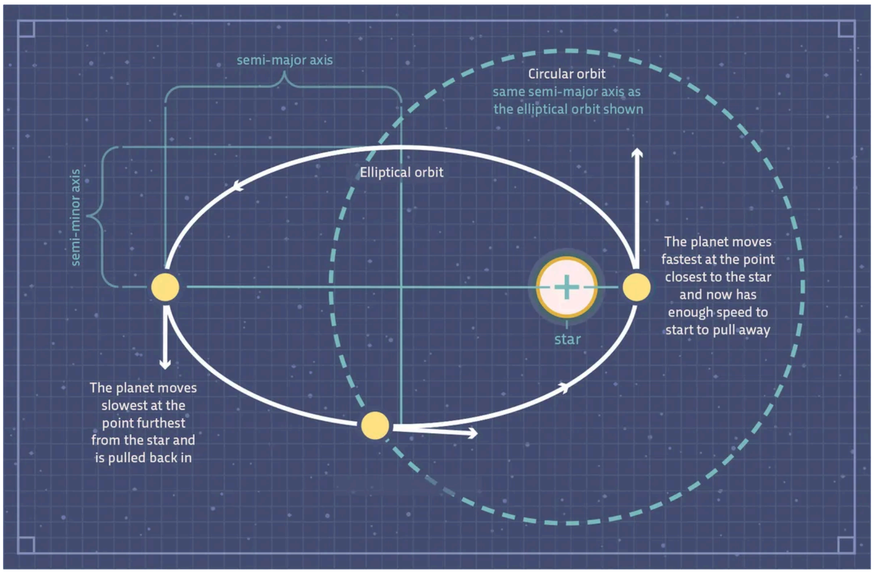
The angular separation of star and planet in arcsec is orbital radius in AU divided by distance in pc.

Answers:
Q0: Earth - 0.67×10^{-9} and Jupiter - 3.82×10^{-9}
Q1: 0.47
Q2:

Body	**Bond albedo, A**	**T**
Mercury	0.088 – reradiates lots	437.9K
Venus	0.76	229.4K
Earth	0.306	254.4K
Moon	0.11 – reradiates lots	270.8K
Mars	0.25	209.9K
Enceladus	0.81 – highly reflective	59.7K
Pluto	0.41	38.9K
Eris	0.99 – highly reflective	10.7K

Q3: 0.1 arcsec.

Section 2: Exoplanetary Orbits

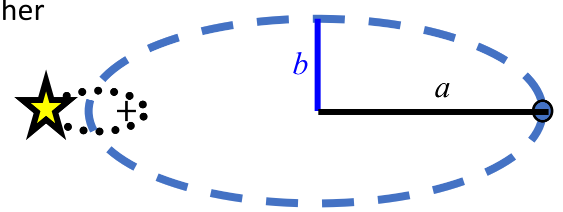


This section covers the topics:

- the common centre of mass,
- the example of Jupiter and the Sun,
- planetary orbits,
- Kepler,
- eccentric orbits,
- orbital angular momentum,
- orbital energy.

Common centre of mass:

A star and its planets orbit their common centre of mass – here we will consider only *a single planet orbiting a single star*. The orbit can be either circular or eccentric. An eccentric orbit is an elliptical one, where the semi-major axis is denoted by *a* and the semi-minor axis is denoted by *b*.



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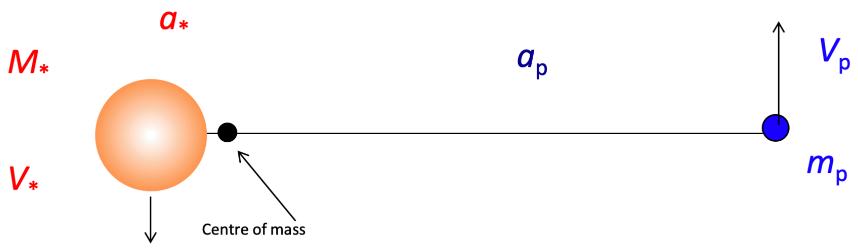
The *reflex orbital velocity* of a star, also known as the radial velocity or Doppler velocity, refers to the speed at which a planet's host star moves in response to the gravitational influence of the planet orbiting around it. This motion about the centre of mass of the system creates a periodic shift in the star's spectral lines due to the Doppler effect, which we can detect. Since this redshift (when the star moves away from us) and blueshift (when the star moves towards us) is periodic, we can use this to infer the presence of an exoplanet.

Let's first think about this for a circular orbit, we want to know what the star's reflex orbital velocity about the centre of mass:
We can write down expressions for the velocities of both the planet and the star using:



$$V_\star = \frac{2\,\pi\,a_\star}{P}, \quad V_p = \frac{2\,\pi\,a_p}{P}$$

Using the conservation of momentum:

$$M_\star\,V_\star = M_p\,V_p \quad \rightarrow \quad M_\star\,a_\star = M_p\,a_p$$

And we can apply Kepler's 3rd law:

$$a^3 = G\,M\left(\frac{P}{2\,\pi}\right)^2 \quad \text{where} \quad M = M_\star + M_p$$

To get an expression for the star's reflex orbital velocity about the centre of mass:

$$V_\star = \frac{2\,\pi\,a_\star}{P} = \frac{2\,\pi}{P}\,\frac{M_p}{M_\star}\,a_p \approx \frac{2\,\pi}{P}\,\frac{M_p}{M}\left(G\,M\left(\frac{P}{2\,\pi}\right)^2\right)^{1/3} = M_p\left(\frac{2\,\pi\,G}{P\,M^2}\right)^{1/3}$$

Notes that this assumes. $M_p << M_\star$ and thus $a_\star << a_p$

Worked example 4:

Jupiter has mass M_J = 0.0009543 M_{sun} and orbits the Sun every P = 11.86 yr. Estimate the Sun's reflex velocity around its centre of mass with Jupiter.

Using Kepler's 3rd Law and noting that $M_p << M_*$:

$$\left(\frac{a}{\mathrm{AU}}\right)^3 = \left(\frac{P}{\mathrm{Year}}\right)^2 \times \left(\frac{M_\star}{M_\odot}\right)$$

$$\Rightarrow a = 11.86^{2/3} = 5.2\ \mathrm{AU} = 5.2 \times 1.496 \times 10^{11}\ \mathrm{m}$$

$$P = 11.86 \times 365.25 \times 86400\ \mathrm{s}$$

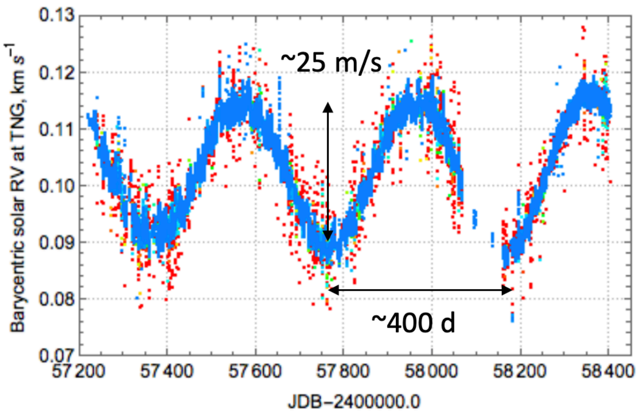
$$V_p = \frac{2\,\pi\,a}{P} = \frac{2\,\pi \times 5.2 \times 1.496 \times 10^{11}}{11.86 \times 365.25 \times 86400} = 1.302 \times 10^4\ \mathrm{m\ s^{-1}}.$$

Then from conservation of angular momentum:

$$V_* = V_p\,\frac{M_p}{M_*} = 0.0009543 \times 1.302 \times 10^4 = 12.46\ \mathrm{m\ s^{-1}}.$$

ASIDE: The Solar reflex motion due to Jupiter:
Whilst we are interested in exoplanets in this course, our own Solar System can be a useful playground. We can observe the Sun's reflex motion around its centre of mass due to the presence of Jupiter, and this is shown in the plot to the right. Calculating the synodic period of Jupiter seen from Earth gives:

$$P_{\rm syn} = \left(\frac{1}{365.25{\rm d}} - \frac{1}{11.86 \times 365.25{\rm d}}\right) = 398{\rm d}$$



Worked example 5:

Estimate the reflex velocity of a 1 solar-mass star with a Jupiter-mass planet in a 1-day orbit:

For Jupiter we found:

$$V_* = V_p \frac{M_p}{M_*} = 0.0009543 \times 1.302 \times 10^4 = 12.46 \text{ m s}^{-1}.$$

In this example our planet has a different period and therefore the planet would have a different velocity, V_p, to Jupiter. Otherwise, everything else is the same so we can scale our previous answer by the ratio of periods to the 1/3 power. This might be obvious to you, but if not, an explanation is given below:

The velocity ratio for our new system is: $\frac{V_*}{V_p} \, for\ new\ system \equiv \frac{V_*}{V_p}$

And the ratio for Jupiter and the Sun from the previous example is: $\frac{V_*}{V_p} \ for\ Jupiter \equiv \frac{V_{*J}}{V_{pJ}}$

And these ratios should equal because $V_* = \frac{V_p M_p}{M_*}$ and the mass ratios are the same for both systems, i.e.

$$\frac{V_*}{V_p} = \frac{V_{*J}}{V_{pJ}}$$

So, the velocity for a star with a Jupiter mass planet in a 1-day orbit would be:

$$V_* = V_p \frac{V_{*J}}{V_{pJ}}$$

The hot-jupiter planet's velocity is:

$$V_p = \frac{2\pi a}{P}$$

And $a \propto (P^2 M)^{1/3}$ so

$$V_p \propto \frac{(P^2 M)^{\frac{1}{3}}}{P} = \left(\frac{M}{P}\right)^{1/3}$$

And we could get the same for Jupiter:

$$V_{pJ} \propto \frac{(P_J^2 M)^{\frac{1}{3}}}{P_J} = \left(\frac{M}{P_J}\right)^{1/3}$$

So

$$V_* = V_p \frac{V_{*J}}{V_{pJ}} = \frac{\left(\frac{M}{P}\right)^{\frac{1}{3}}}{\left(\frac{M}{P_J}\right)^{1/3}} V_{*J} = \left(\frac{P_J}{P}\right)^{1/3} V_{*J}$$

i.e. we can scale our previous answer by the ratio of periods to the 1/3 power:

$$V_* = 12.46 \times (11.86 \times 365.25)^{1/3} = 203.2 \text{ m s}^{-1}$$

Hot Jupiters (small P) are a lot easier to detect using radial-velocity searches than cold ones!

Kepler's laws of planetary motion:

1. **Kepler's First Law (Law of Ellipses):**
Kepler's first law states that the orbit of a planet around the Sun is an ellipse, with the Sun at one of the two foci. In other words, planets do not move in perfect circles but rather in elongated orbits where the Sun is slightly off-center.

2. **Second Law (Law of Equal Areas):**
Kepler's second law, also known as the law of equal areas, states that a line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time. This means that a planet travels faster in the part of its orbit where it is closer to the Sun and slower when it is farther away.

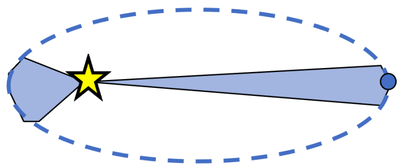
3. **Third Law (Harmonic Law):**
Kepler's third law establishes a relationship between the orbital period of a planet, *P*, and the semi-major axis of its orbit, *a*. It can be expressed as:

$$P^2 \propto a^3$$

i.e., the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

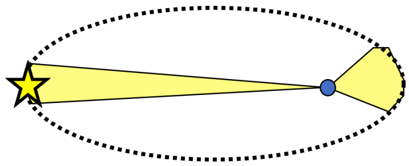
A reminder about reference frames:
Last semester you had a lot of practice thinking about reference frames in the Special Relativity course. Hopefully you remember that reference frames are not specific to special relativity and are common in classical mechanics too. Reference frames are different ways of thinking about a problem. In the case of a planet orbiting a star, this could be from the point of view of the star (where the star is stationary), from the point of view of the planet (planet is stationary) or from *the inertial frame*.

Star's view:



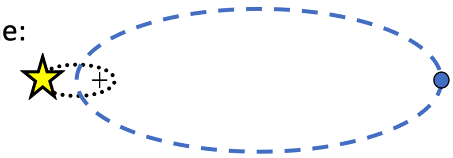
The planetary orbit is an ellipse with the star at one focus. The planet sweeps out equal area in equal time (angular momentum conservation).

Planet's view:



The planet is stationary at the focus and the star sweeps equal area in equal time.

Inertial Frame:

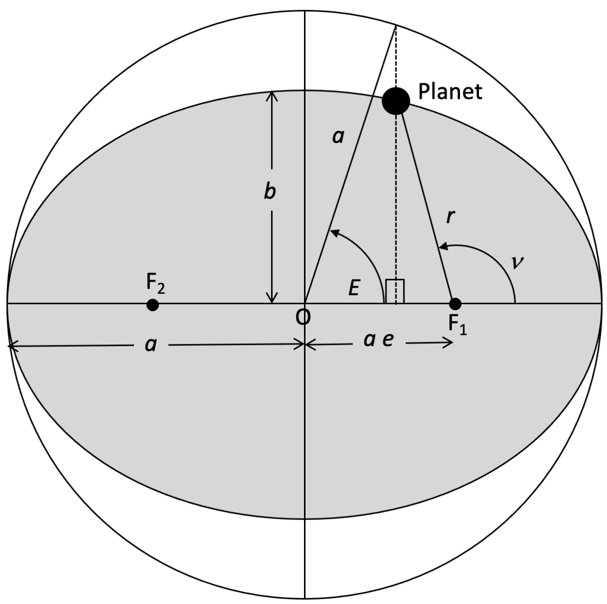


The star and planet both orbit around the centre of mass.

Below are some useful relations for Keplerian orbits:

$$M = M_\star + M_p = \text{ total mass}$$
$$a = a_p + a_\star = \text{ semi} - \text{major axis}$$
$$a_p\, M_p = a_\star\, M_\star \text{ centre of mass}$$
$$P \;= \text{orbit period}$$
$$a^3 = G\, M\left(\frac{P}{2\,\pi}\right)^2 \text{ Kepler's 3rd Law}$$
$$e \equiv \sqrt{1-(b/a)^2} = \text{ eccentricity}$$
$$0 < e < 1 \quad 0 = \text{circular} \quad 1 = \text{parabolic}$$

<u>The anatomy of an eccentric orbit:</u>



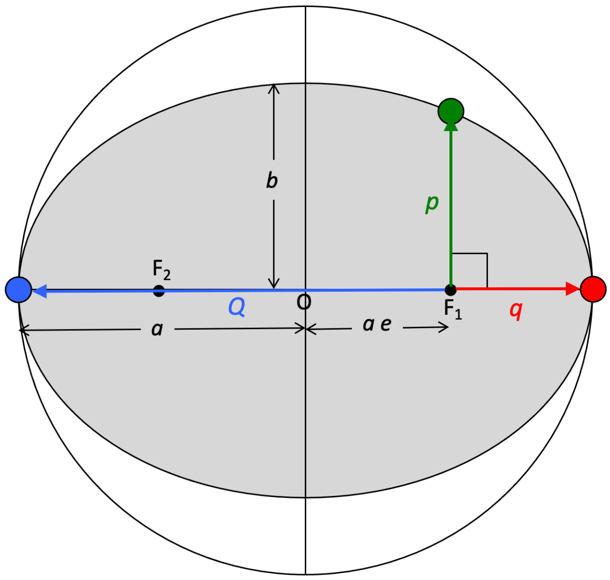
The image below shows an eccentric orbit where the symbols have the following meanings:

- F_1 and F_2 are the focal points of the ellipse, and the star is at F_1.
- Semi-axes *a, b*
- Eccentricity *e*: $b^2 = a^2(1-e^2)$
- Area of ellipse $A = \pi ab$
- Eccentric anomaly *E*
  Uses *auxiliary circle*
- True anomaly ν
  Periastron-Star-Planet
- Planet-star distance
  Equation of an ellipse

$$r = \frac{a(1-e^2)}{1+e\cos\nu}$$

Whilst you are not expected to memorise these equations, it is important that you know what all the terms in the equations are so that you can then apply these equations when you need to.

Q0: Can you derive the expression for the planet-star distance, *r*, using trigonometry, given that the "string length" F1 to P to F2 is constant?



Using $r = \dfrac{a(1-e^2)}{1+e\cos\nu}$ we can calculate the planet-star distance at a few important points in the orbit:

- Periastron (ν = 0):

$$q = \frac{a(1-e^2)}{1+e} = a(1-e)$$

- Apastron (ν = 180°):

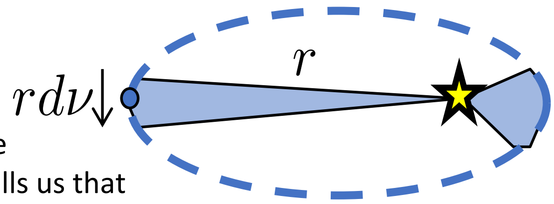
$$Q = \frac{a(1-e^2)}{1-e} = a(1+e)$$

- Semi-latus rectum (ν = 90°):

$$p = \frac{a(1-e^2)}{1+0} = a(1-e^2)$$

Q1: can you convince yourself that you can get these expressions for periastron, apastron and semi-latus rectum given the true anomaly values given?

<u>Kepler's 2nd law (again):</u>



A planet moving through a small angle dν sweeps out area (1/2) r^2dν in time *dt.* Since this is a small angle, we can treat the shaded blue areas as triangles. Kepler 2 tells us that the planet sweeps out equal areas in equal times.
Over a full orbit (in time *P*) the planet sweeps out the full area of the ellipse, so the rate at which area is swept out is:

$$\frac{1}{2}r^2\frac{d\nu}{dt} = \frac{\pi ab}{P} = \frac{\pi a^2(1-e^2)^{1/2}}{P}.$$

<u>Orbital angular momentum:</u>

The planet's transverse velocity component is: $v_\perp = r\dfrac{d\nu}{dt}.$

The angular momentum per unit mass is given by

$$L = |\underline{r}\times\underline{v}| = rv_\perp = r^2\frac{d\nu}{dt} = 2\frac{\pi a^2(1-e^2)^{1/2}}{P}.$$

We can substitute for *P* using Kepler 3:

$$P = \sqrt{\frac{4\pi^2 a^3}{GM}}$$

To get the angular momentum of the orbit:

$$L = \sqrt{GMa(1-e^2)} = \sqrt{GMp}$$

Where $p = a(1-e^2)$ is the semi-latus rectum of the orbit.

<u>Orbital Energy:</u>

The orbital energy per unit mass is

$$E_{\rm tot} = E_{\rm grav} + E_{\rm kin} = -\frac{GM}{r} + \frac{1}{2}v^2.$$

At periastron:

$$r = a(1-e) \Rightarrow E_{\rm grav} = -\frac{GM}{a(1-e)}.$$

$$E_{\rm kin} = \frac{1}{2}v^2 = \frac{1}{2}\frac{L^2}{r^2} = \frac{GMa(1-e^2)}{2a^2(1-e)^2} = \frac{GM(1+e)}{2a(1-e)}.$$

The total energy (here calculated at periastron) is:

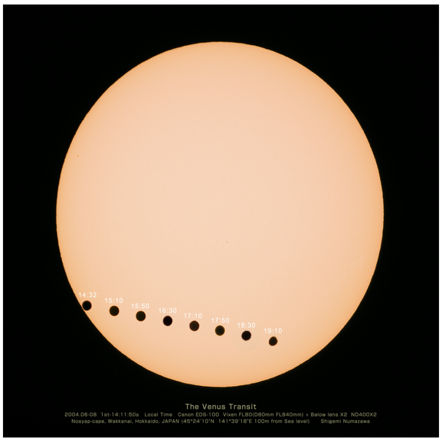
$$E_{\rm grav} + E_{\rm kin} = -\frac{GM}{a(1-e)} + \frac{GM(1+e)}{2a(1-e)} = -\frac{GM}{2a}.$$

Q2: show that you get the same answer at apastron where *r = Q = a(1+e).*

Section 3: Hunting exoplanets – The Transit Method

The simplest method to find planets is to look for the drop in stellar flux due to a planet transiting across the stellar disc.

We can observe planets within our Solar System transiting the Sun in the same manor – Venus is shown in the photo to the right. We don't observe these transits every time the planet passes between the Sun and Earth though. The reason for this is that Venus and Earth's orbits are slightly inclined compared to each other. We need the geometry to work out for Venus to transit the Sun from our perspective.

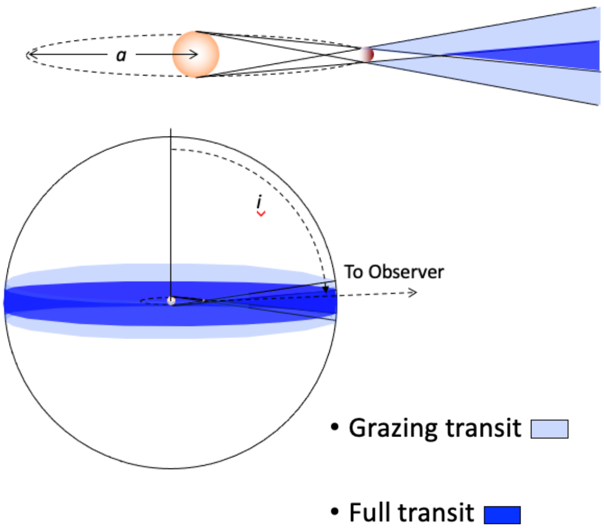


Transits only occur if the orbit is almost edge-on. We might want to know what the probability of observing a transit might be.

We can classify transits into two groups (i) full transits and (ii) grazing transits:

Full transits occur when the whole planet crosses the stellar disc, whilst grazing transits occur when only a part of the planet crosses the stellar disc.

As the planet orbits the star it creates a region of space that is within the planet's shadow. We can then draw a 'shadow band' in a sphere of the parts of the sky that the planet can be in and cause a transit.



Mathematically we can describe grazing transits by:

$$\left| a\cos i \right| \leq R_* + R_p$$

And full transits by:

$$\left| a\cos i \right| \leq R_* - R_p$$

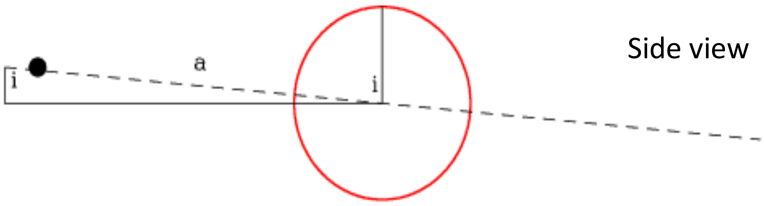
These occur when the whole planet (R_p) fits within the stellar disc (R_*).

We might then ask, "in what kinds of system are you most likely to see a transit?"

You are more likely to see a transit from:

- a planet close to its host star (small *a*),
- inclinations (*i*) that mean you look into the plane of the planetary orbit,
- planets with large R_p/R_*.

Side view

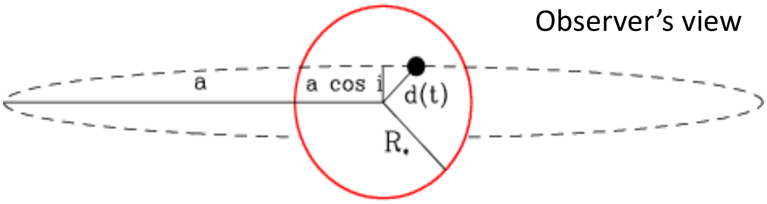


<u>Transit probabilities:</u>

Transits occur only in nearly edge-on orbits:

$$|a\cos i| \le R_* + R_p$$

Observer's view



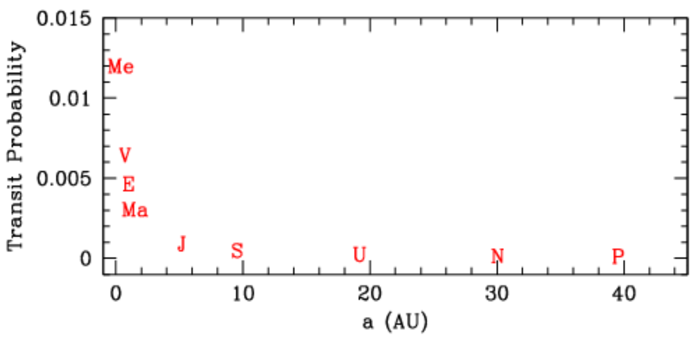
The transit probability is given by:

$$\text{Prob}\left(|\cos \text{i}| < \frac{R_* + R_p}{a}\right) = \frac{R_* + R_p}{a} \approx \frac{R_*}{a}$$

where we have assumed that the radius of the planet is much smaller than the radius of the star.

Q0: What does this expression tell us about the kinds of planets that we can find by the transit method?

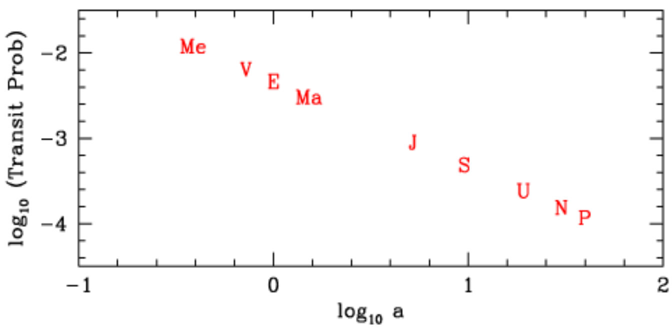
Transit probabilities within our own Solar System:



We can scale the probability of a transit expression to our own solar system to get:

$$\text{Prob} \approx \frac{R_*}{a} \approx 0.005\left(\frac{R_*}{R_{sun}}\right)\left(\frac{1AU}{a}\right)$$

Q1: Can you get this expression for the scaling relation?



We can see that hot planets are more likely to be detected than cold ones. This expression tells us that the probability at 1AU is 0.5% i.e. the probability of seeing Earth transit the Sun.

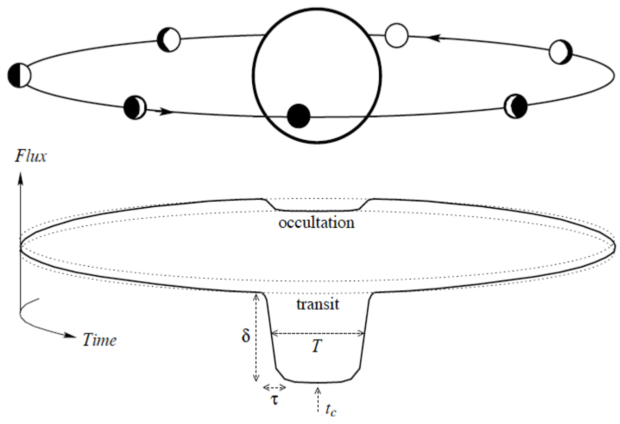
Meanwhile, the probability at 5AU (Jupiter's orbit) is:

$$Prob = 0.005\left(\frac{R*}{Rsun}\right)\left(\frac{1AU}{a}\right) = 0.005 \times 1 \times \frac{1}{5} = 0.1\%$$

For hot jupiters – 0.05AU:

$$Prob = 0.005\left(\frac{R*}{Rsun}\right)\left(\frac{1AU}{a}\right) = 0.005 \times 1 \times \frac{1}{0.05} = 10\%$$

So, thousands of stars must be monitored to discover planets by spotting their transits.



Q3: If we move a planet further away from the star, what happens to the transit depth?

The transit depth is dependent on the fraction of the star's disc that the planet covers up:

$$\frac{\Delta f}{f} \approx \left(\frac{R_p}{R_\star}\right)^2 = 0.01\left(\frac{R_p}{R_{\rm Jup}}\right)^2\left(\frac{R_\star}{R_\odot}\right)^{-2}$$

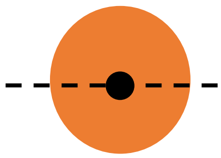
The observed depth tells us about the planet to star area ratio and we can find the stellar radius from its spectral type.

Q4. What fraction of a host star would the following planets cover: Venus at 0.72AU and Pluto at 39.5AU?

Q5: If we move a planet further away from the star, what happens to the **transit duration**?

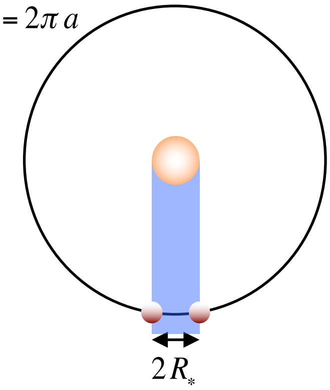
<u>Transit duration gives the stellar density:</u>

Let's look at the simplest case of a circular orbit and i = 90 degrees – i.e. the observer looks directly into the plane of the transit.



Now imagine looking down on the system.

The planet traces out a circle with circumference = $2\pi a$
The transit occurs over the distance (approx.) $2R_*$



So

$$\frac{T}{P} \approx \frac{2R_*}{2\pi a}$$

And using Kepler's third law:

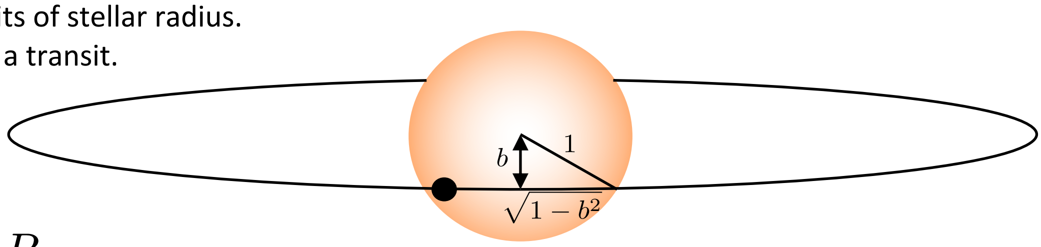
$$a^3 = GM_*\left(\frac{P}{2\pi}\right)^2$$

And using $\rho_\star = \frac{3}{4\pi}\frac{M_\star}{R_\star^3}$ gives:

$$\frac{T}{P} \approx \frac{R_*}{\pi a} = \frac{R_*}{\pi}\left(\frac{4\pi^2}{GM_* P^2}\right)^{1/3}$$

$$T \approx 3\left(\frac{P}{4\text{ days}}\right)^{1/3}\left(\frac{\rho_*}{\rho_{Sun}}\right)^{-1/3} \text{ hours}$$

But what if our orbit isn't edge on to the observer?
Real orbits have non-zero inclinations and so we define the **impact parameter b**. This parameter is the projected distance of the planet from the centre of the star at mid-transit and is measured in units of stellar radius.
So, $0 < b < 1 + R_p/R_*$ for a transit.



$$\frac{T}{P} \approx \frac{\sqrt{1-b^2}}{\pi}\frac{R_*}{a}$$

So, what happens to the transit duration if the transit is not edge on?

- Higher b means smaller T/P.
- Transit duration decreases – makes sense, planet doesn't have to cross as much of the disc so it shouldn't take as long!

<u>Planetary surface gravity:</u>
We can calculate the surface gravity of a planet from the observations!
Imagine now that we also go out and find the planet via radial velocity:
We still are considering our simple case with a circular orbit, and edge-on view (i = 90°).
The stellar reflex velocity is given, in this case, by

$$V_\star = K \sin\left(\frac{2\pi(t - T_0)}{P}\right)$$

This gives the star's radial velocity curve, and we can find the slope of this using:

$$\frac{dV_\star}{dt} = \frac{2\pi K}{P} = \frac{G M_p}{a^2} = g_p \frac{R_p^2}{a^2}$$

The planet provides the gravitational pull on the star and causes the wobble therefore this $\frac{2\pi K}{P}$ has to equal $\frac{GMp}{a^2}$ i.e. the star's orbital acceleration = slope of RV curve at t = T_0:

$$\frac{2\pi K}{P} = g_p \left(\frac{R_p}{R_*}\right)^2 \left(\frac{R_*}{a}\right)^2$$

Stellar radial acceleration at conjunction

Planet surface gravity

Transit depth

Transit duration

With this equation we can calculate the surface gravity of our observed exoplanets! From this, we can then get the planetary density:

$V = \frac{4}{3}\pi R_p^3$ and $\rho = \frac{M}{V} = \frac{3M_p}{4\pi R_p^3}$

Insert the planet surface gravity:

$$g_p = \frac{G\,M_p}{R_p^2}$$

To get the density:

$$\rho = \frac{3\,g_p}{4\,\pi\,R_p\,G}$$

We have R_p / R_* from the transit depth, so, to find the density, we need to measure R_* so that we can get R_p from R_p / R_* . One can do some clever stuff, if so inclined, using new data in the Gaia catalogue involving the angular radius of the star and the parallax to get the stellar radius and in turn get an expression for the planetary radius:

$$R_p = 1 \text{ AU} \times \frac{\theta}{2\,\hat{\pi}} \left(\frac{R_p}{R_*}\right)$$

Putting it all together, the planet's bulk density is found purely from observable quantities as

Stellar orbital velocity amplitude

Stellar parallax (GAIA)

$$\rho_p = \frac{3K}{a_\oplus GP}\,\frac{\hat{\pi}}{\theta} \left(\frac{R_*}{R_p}\right)^3 \left(\frac{a}{R_*}\right)^2$$

Earth orbit radius = 1 AU

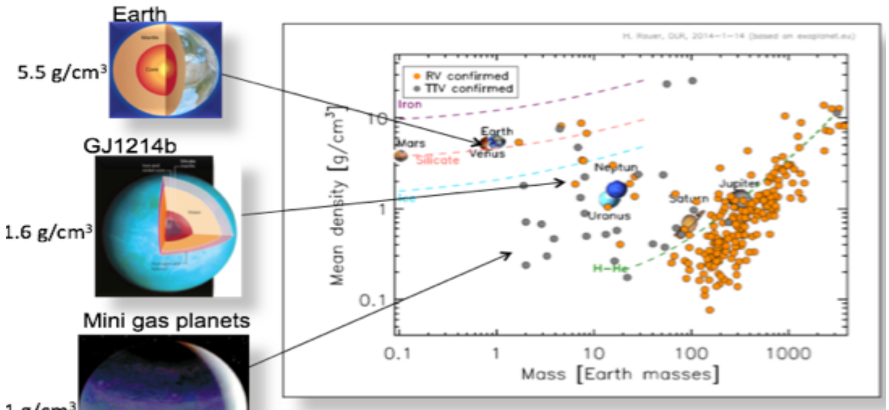
Planet orbital period

Stellar angular diameter (T_{eff} and f_{bol})

Transit depth

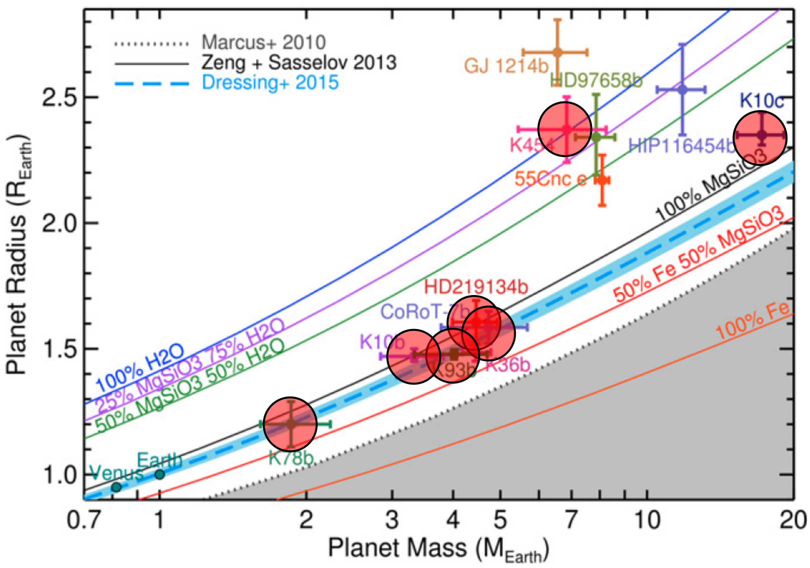
Transit duration

(Clearly this is not an expression to memorise! But it shows you how all the observed quantities can be combined and used to calculate the planetary density). From this we find that there is a lot of diversity in the structure of exoplanets and this can be shown pictorially in the plot below:



Low-mass planets have a range of compositions and interior structures for similar masses.

We can also make a similar plot showing contours of constant composition:



We see lots of lower mass planets seem to have compositions similar to Earth, but for larger planets there's a big jump up to less dense composition.

Answers:

Q0: Transit surveys find planets in small orbits around large parent stars.

Q1: $Prob = \frac{R*}{a} = \left(\frac{R*}{Rsun}\right)\left(\frac{1AU}{a}\right)\left(\frac{Rsun}{1AU}\right) = 0.005\left(\frac{R*}{Rsun}\right)\left(\frac{1AU}{a}\right)$

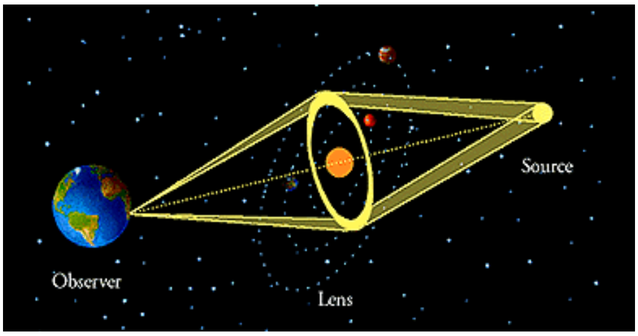
Q3: the transit depth will decrease.

Q4. Venus: 7.5×10^{-5}, Pluto: 2.9×10^{-6}

Q5: It should increase – it will take longer for the planet to track across the stellar disc.

Section 4: Hunting exoplanets – Microlensing

Gravitational microlensing is the hardest way to find exoplanets, we can think of it a bit like "looking for (and finding) invisible planets around invisible stars".
But this technique is particularly useful because it isn't biased to very close-in planets like for transits or radial-velocity! Therefore, it is effective for finding cool planets.



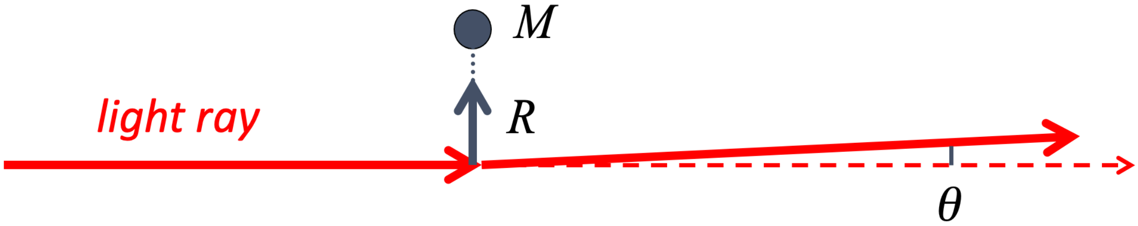
The sweet spot for finding exoplanets by this method is a few AU from the host star. And generally seen around a low mass star (Mdwarf) just because there are very many more of them in the Universe! So, an example of a "good" system for microlensing would be:

$$M_{\text{Lens}} \sim 0.3\, M_{\text{Sun}} \qquad R_{\text{E}} \sim 3\,\text{AU} \sim 10^{-3}\,\text{arcsec}$$

Gravitational microlensing was predicted by Einstein's general relativity – although Einstein himself thought we would never see it! There are two aspects of general relativity that that are useful to our understanding of the phenomenon of microlensing:

- particles (and light) follow shortest available paths (aka "geodesics") through spacetime,
- and mass (and energy) causes spacetime to warp.

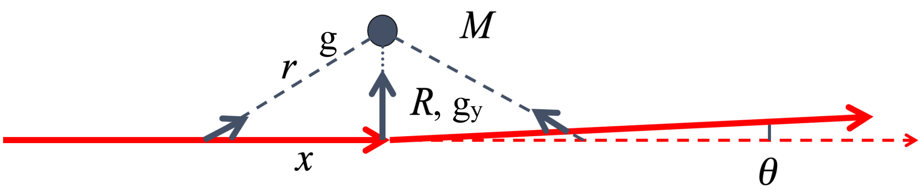
We can draw light moving around a massive object as:



(noting that light travels in a straight line through curved space, but that is a bit tricky to draw!)
Einstein calculated this θ or "bend angle" using general relativity and found:

$$\text{Einstein's bend angle} \qquad \theta = \frac{4\,G\,M}{R\;c^2}$$

We can use some Newtonian physics to estimate this bend angle.



Start by writing the vertical acceleration as g_y. Then notice that the fraction, g_y/g must equal R/r. This then gives us an expression for g_y which is essentially an acceleration. Since this is an acceleration, we can obtain an expression for the vertical velocity.

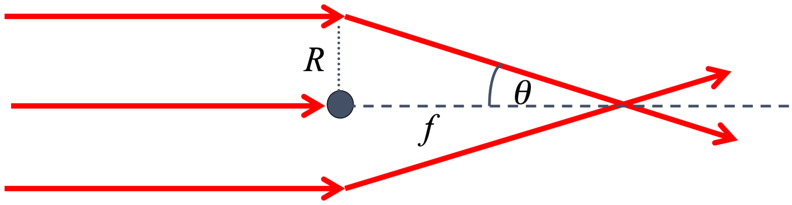
$$\text{vertical acceleration} \quad g_y = \left(\frac{G\,M}{r^2}\right)\left(\frac{R}{r}\right) \qquad r^2 = R^2 + x^2$$

$$\text{vertical velocity} \quad V_y = \int_{-\infty}^{\infty} g_y\,dt = \int_{-\infty}^{\infty} \frac{G\,M\,R}{\left(R^2 + x^2\right)^{3/2}} \frac{dx}{V_x} = \frac{2\,G\,M}{R V_x}$$

$$\text{bend angle} \qquad \theta \approx \frac{V_y}{V_x} \approx \frac{2\,G\,M}{R\,V_x^2} \Rightarrow \frac{2\,G\,M}{R\,c^2}$$

Noticing that the x-component of the velocity is c, we can obtain an expression for the bend angle with a little algebra. Remarkably, this expression is very similar to the true expression for the bend angle – what is a factor of 2 between friends?!

If we now consider not just one light beam passing this lens star but multiple, we see that the black dot (our lens star) truly does act like a lens (diagram below!). Knowing the bend angle, we can obtain an expression for the focal length of this space-lens.

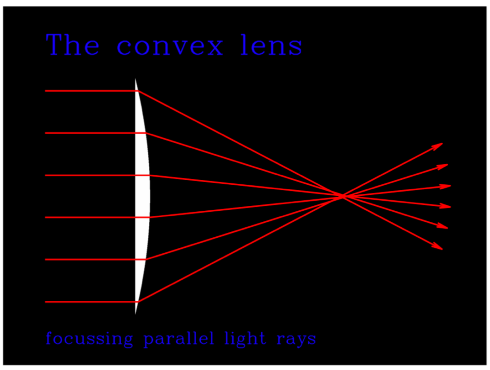


$$\text{Einstein's bend angle} \qquad \theta = \frac{4\,G\,M}{R\;c^2}$$

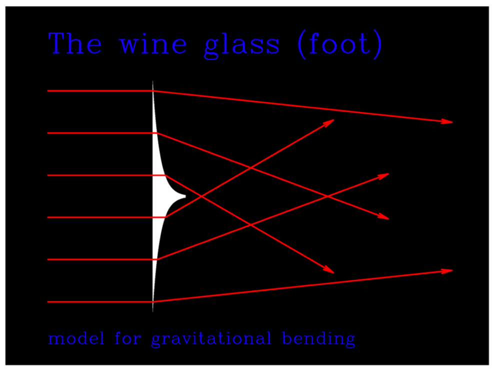
$$\text{Focal length:} \quad f = \frac{R}{\theta} = \frac{R^2\,c^2}{4\,G\,M}$$

But hold up! This focal length depends on R. What does this tell us?

Light from different distances around the object focuses differently!



versus

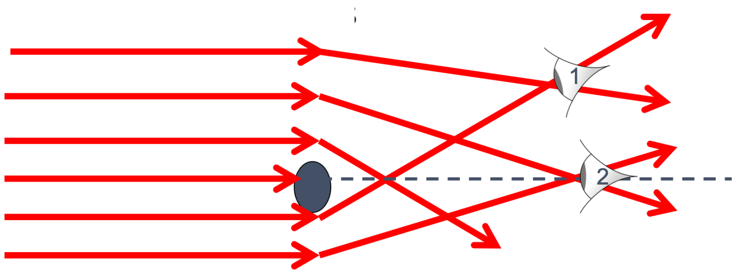


Credit: http://www.artemis-uk.org/Microlensing_physmath.html

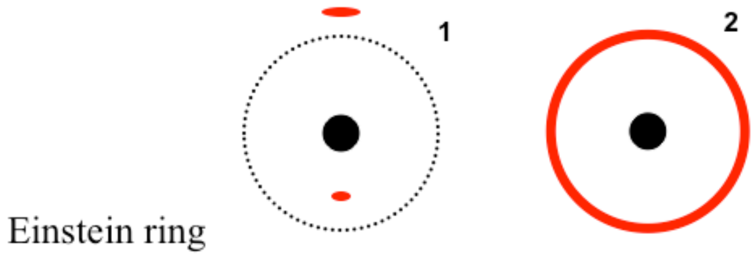
The image above on the left shows a lens that we are used to. Light comes into the lens and gets focussed to some point a focal length away from the lens. The image on the right shows the kind of lensing that occurs with microlensing.

Lensing by a point mass:

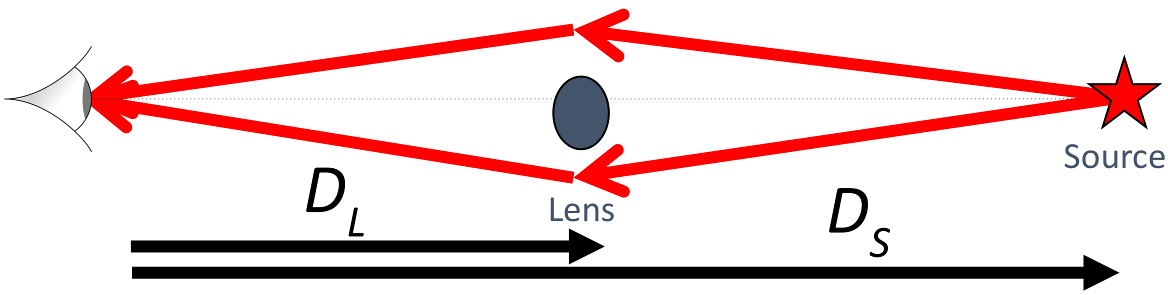
Light from a background source deflected by lens mass (e.g. star)



And two distorted/magnified images of the background source are created:



If viewed head-on (2), an Einstein ring is seen, but in most cases, we observe two spots above and below the lens object. We can calculate the radius of the Einstein ring by using the lens makers equation and a bit of optics…



Geometric optics :

$$\frac{1}{f} = \frac{1}{D_S - D_L} + \frac{1}{D_L} = \frac{4GM}{c^2 R^2}$$

$$\Rightarrow \frac{1}{D_S(1-X)} + \frac{1}{XD_S} = \frac{4GM}{c^2R^2} \text{ where } X \equiv \frac{D_L}{D_S}$$

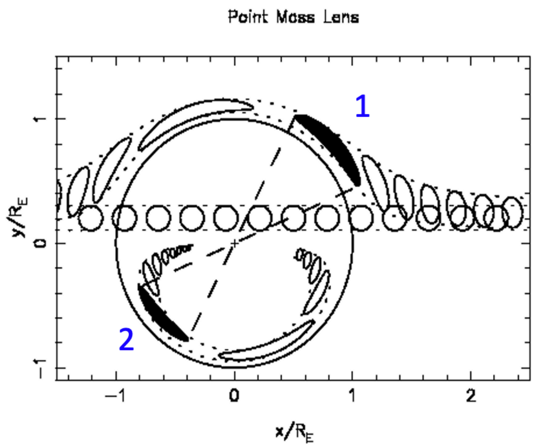
Rearrange to get Einstein Ring radius : $R = R_E$

$$R_E = \left(\frac{4GM}{c^2}\right)^{1/2} D_S^{1/2}\sqrt{X(1-X)} = 8\mathrm{AU}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_S}{8\mathrm{kpc}}\right)^{1/2}\sqrt{X(1-X)}$$

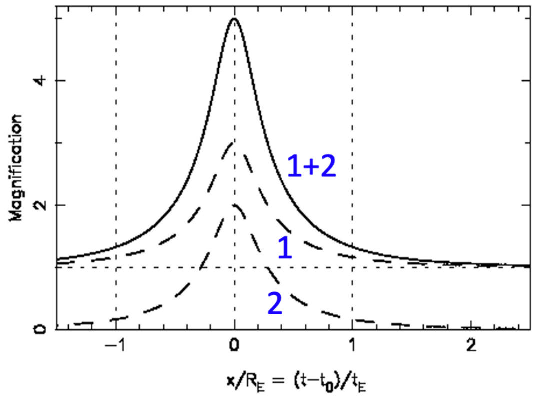
$$\theta_E = \frac{R_E}{D_L} = \sqrt{\frac{4GM}{c^2}\left(\frac{1}{D_L} - \frac{1}{D_S}\right)} \simeq 10^{-3}\mathrm{arcsec}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_S}{8\mathrm{kpc}}\right)^{-1/2}\sqrt{\frac{1-X}{X}}$$

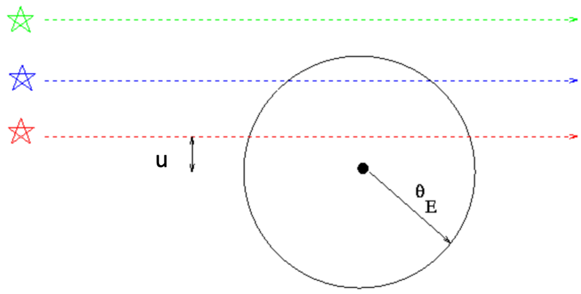
Lensing can happen by galaxies and galaxy clusters, but we care about lensing of stars by stars, and especially in the microlensing caused by any associated exoplanets.

With a point mass lens, 2 images are seen on opposite sides of lens. There is one major image outside the ring (1 in the image) and a minor image inside ring (2 in the image). The magnification, which we denote as *A(t)*, is the sum of the areas of the two images and changes with time. This magnification varies with time, getting brighter as the source object reaches the centre of the Einstein ring, before decreasing again.



The magnification, *A,* depends on the impact parameter, $u = R / R_E$. For a small impact parameter (e.g. red in the cartoon below), there is a large magnification and source stars with a larger impact parameter (blue) would have a lower magnification.





Red – brightest
Blue – not so bright

Microlensing is an achromatic phenomenon, i.e. all wavelengths are affected equally, and the chances of microlensing are ~ 1 in a million for the lensing of Milky-Way stars at D_S ~ 8 kpc (i.e. not very high!). For any given star, it will be lensed only once.

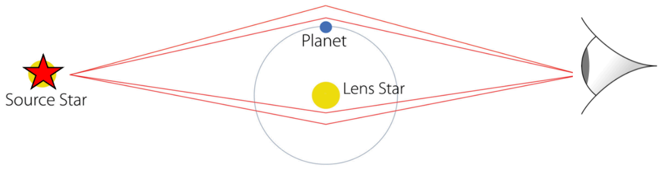
Finding planets using microlensing:
When the lens star has an orbiting planet, the lens system is a **binary lens** with an extremely small mass ratio m_p / M_*. Most binary lens events are binary stars, and not a star-planet system. Binary lens light curves are complicated, with multiple peaks and sharp jumps when new image pairs emerge or disappear. With good monitoring, the parameters of the binary can be recovered, and **we can determine the star and planet mass, the orbit size and the distance.**

To discover exoplanets by this method, observers use the following strategy:

- Monitor **many** (~10^9) stars to discover microlensing events (finding ~2000 stellar events/yr).
- Monitor the most promising events with dense, high precision photometry ***from several sites***.
- Look for deviations from single-star light curve due to planets.

This requires lots of time on small telescopes around the world.

The timescales for the planetary blips seen are ~ few days for Jupiters, ~ few hours for Earths and the method is most sensitive to planets with $a \sim R_E$, near Einstein ring radius (~ 3-5 AU for typical parameters).

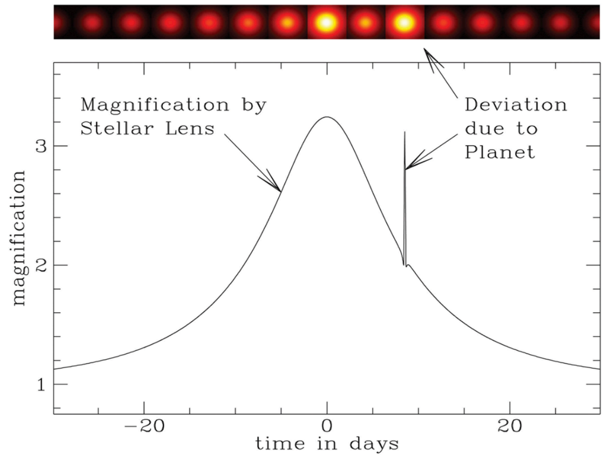


Microlensing planet light curves:

The planets act as a smaller-mass lens and within the light curve there is a large (but brief) signal from small planets. The duration of the whole event signal scales as Einstein Ring radius:

$$t_E \propto R_E \propto \sqrt{M}$$

Which scales as the mass of the star. The duration of the planetary spike compared to the main event depends on the mass ratio, m_p / M_*.



Summarising microlensing:

- 278 planets have been found so far by microlensing*, in 256 systems and including 10 multi-planet systems. Earth-mass planets can be detected – the smallest three are 1.4, 1.7, and 2.3 Earth masses.
- Typical lens stars are ~0.3 M_{sun} (0.1 - 2 M_{sun})
- It is sensitive to “cool” planets at ~ 3-5 AU - outside the “Snow Line”.
- Planet detections are sometimes ambiguous - one light curve can have several possible solutions for the properties of the source.
- There are **no repeat observations** – it is a one-time event.

Some potentially useful resources:

- https://www.youtube.com/watch?v=_aZZt8dM-_0
- https://www.youtube.com/watch?v=k7U_RGDWP94
- (*) https://exoplanet.eu/catalog/

Some visualisations:

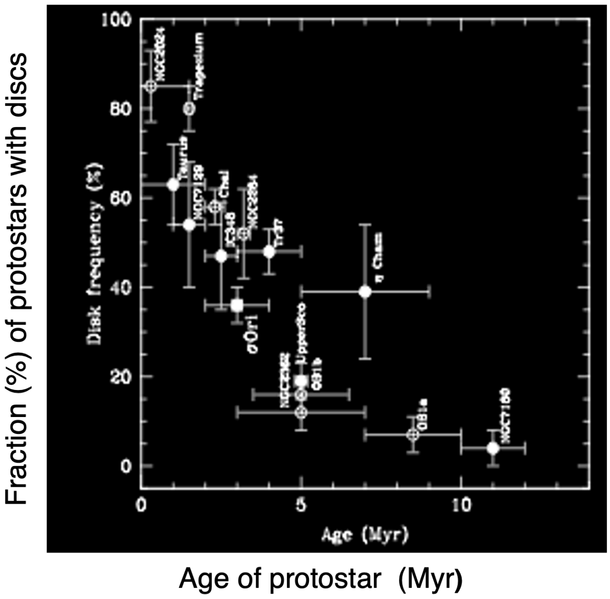
- https://science.nasa.gov/mission/roman-space-telescope/microlensing/
- https://www.youtube.com/watch?v=Wkf3AkVvPAM
- https://www.youtube.com/watch?v=d2XZYFWPWiE

Section 5: So, you want to build a planet?

We know that planets form within protoplanetary discs, but how?

Protoplanetary discs are 99% gas by mass (specifically, about 75% H, 24% He, <1% others). These discs are 1% dust grains by mass i.e. solids (graphite, silicates) and ices (H_2O, CO, CO_2,…).
They have a typical size of: 10s-100s AU

It would be easy to make planets if you had infinite time… but within our discs, most of the gas has dissipated by 10Myrs (see plot to the left). This means that we must form planets faster than this, or else there won't be anything left to make them from!

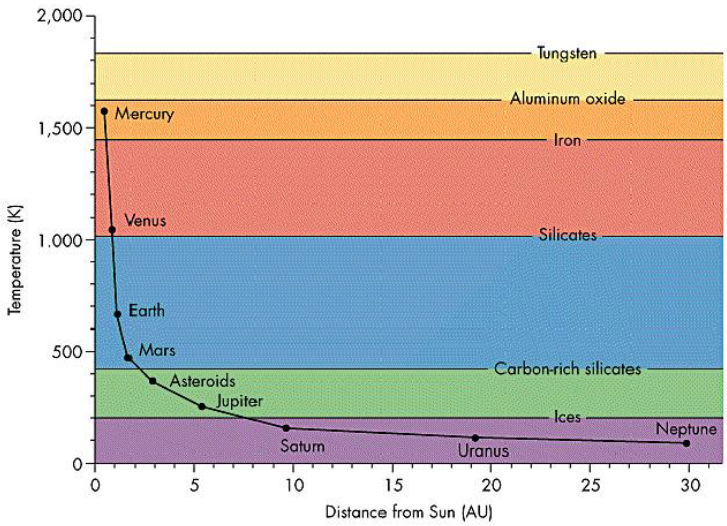


Imagine that we ground up all the planets in the solar system and then distributed the mass smoothly with radius. Then we add back enough gas (H, He) to restore the solar composition. This way, we have made a disc that should be the same as the one that the Sun had in its youth.

We could calculate a thing for our disc called "a surface density profile". This tells us about how the mass is spread out over the area of the disc, and we could calculate it with the following expression:

$$\Sigma(R) \approx 10^4 \left(\frac{R}{1\ \mathrm{AU}} \right)^{-3/2} \quad \mathrm{kg\ m^{-2}}$$

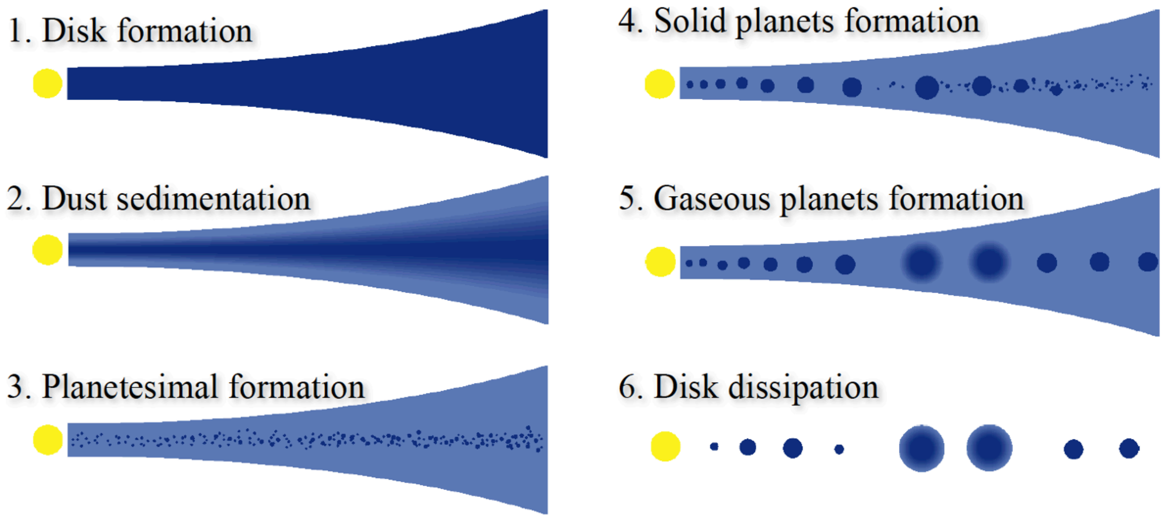
Our disc would have a total mass of ~0.01 M_{Sun} out to ~30 AU.



With this information about the surface density, we could combine information about the heating and cooling of the disc by friction and stellar irradiation, and this would enable us to compute:

- The temperature structure *T (R)* of the disc,
- And therefore the "snow Lines".
  These are the distances outside of which different minerals and ices "freeze out" and become solids.

The key stages in planet formation:



The image about shows different steps in the planet making process. The four main stages of planet formation are given in the table below, alongside the role of the protoplanetary disc gas and the timescale over which the stage happens.

Stage	Role of gas	Time scale
1. Settling and growth of dust grains	Dust well-coupled to gas	Probably rapid.
2. Pebbles and boulders to km-sized planetesimals	Hydrodynamic drag -> inward motion of the solids	Must be rapid.
3. Planetesimals to planet-sized bodies/giant-planet cores	Independent of gas.	Slow.
4. Gas accretion on to giant-planet cores	Orbital migration due to gravitational torques.	Slow.

Let's talk a bit about disc structure and then each of these stages of planet formation.

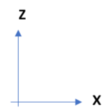
Disc structure:

If we want to know about the structure within the disc, it would be sensible to start by considering how its density varies with height above the midplane. To do this, we need to consider the forces acting on the gas. The three forces are the centrifugal force, the gravitational force and the gas pressure.

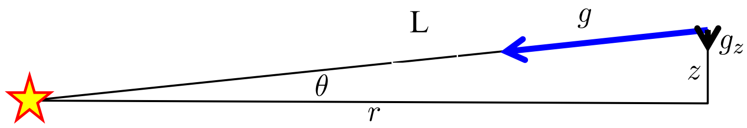
Gravitational force from the centrifugal force:

We can calculate the gravitational acceleration with height above the midplane of the disc:

Lucy Hadfield's voice is ringing out in your head "first you draw your picture, and then you draw your coordinate system".



The z component of the gravitational acceleration can be written down using some trig.



$$\frac{g_z}{g} = \frac{z}{L} = \frac{z}{\sqrt{r^2+z^2}}$$

$$g_z = g\frac{z}{\sqrt{r^2+z^2}} = \frac{GM_*}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}}$$

And then we note that we have a thin disc (so z << r) and that it is Keplerian (so $F_g = F_{cent}$ so we can write $g \approx \Omega^2 r$). With this information, we can write down the following:

Keplerian:

$$g = \frac{GM_*}{r^2+z^2} \approx \Omega^2 r$$

Thin:

$$\frac{z}{\sqrt{r^2+z^2}} \approx \frac{z}{r}$$

And combining:

$$g_z = \frac{GM_*}{r^2+z^2}\frac{z}{\sqrt{r^2+z^2}} \approx \Omega^2 r\frac{z}{r} = \Omega^2 z$$

This has given us an expression for how the vertical component of the acceleration varies with height (z).

Gas pressure from the gravitational force:

Hydrostatic equilibrium tells us that the pressure and gravitational forces in our disc are balanced. Mathematically we would write this as:

$$\boldsymbol{\nabla} P = \rho \boldsymbol{g}$$

This must be truly vertically, in other words, the force from the gas pressure pushing up is balanced by the gravitational force pulling down.

$$\frac{dP}{dz} = -\rho_g g_z$$

Where ρ_g is the gas density within the disc and P is the gas pressure.

But we can't integrate this as it is to get an expression for the gas density. Pressure and density are related to each other (remember $PV = nRT$ from high school?). The pressure term P depends on the density, so in order to integrate this we need to write the pressure in terms of density explicitly.

We can do this by using the sound speed in the gas, which can be written using:

$$c_s^2 = \frac{P}{\rho_g} (= \frac{K_B T}{\mu m_H} \; for \; an \; isothermal \; gas)$$

We can then eliminate pressure from the hydrostatic equilibrium expression:

$$\frac{dP}{dz} = -\rho_g g_z$$

$$\frac{d(c_s^2 \rho_g)}{dz} = -\rho_g g_z$$

$$c_s^2 \frac{d\rho_g}{dz} = -\rho_g g_z$$

(because the speed of sound is constant with z, so we can pull it out of the differential) and using the expression we got for the vertical component of gravity:

$$c_s^2 \frac{d\rho_g}{dz} = -\rho_g \Omega^2 z$$

We can then integrate this to solve for the density as a function of height:

$$\frac{1}{\rho_g} \frac{d\rho_g}{dz} = -\frac{\Omega^2 z}{c_s^2}$$

$$\int_{\rho(0) \equiv \rho 0}^{\rho(Z)} \frac{d\rho_g}{\rho_g} = \int_0^Z -\frac{\Omega^2 z}{c_s^2} dz$$

Our integral limits are easy for the right-hand side, from 0 (the midplane) up to some arbitrary height *Z* (Note: you could of course keep this labelled as *z*, but it is bad practice to use the same label for a limit as for your variable (*dz*)!) For the left-hand side, the density varies from the density in the midplane $\rho(0)$, that we chose to call $\rho 0$, to the density at the height *Z*, $\rho(Z)$.

$$\ln\big(\rho_g(Z)\big) - \ln(\rho_0) = -\frac{\Omega^2}{c_s^2} \int_0^Z z \, dz = -\frac{\Omega^2}{c_s^2} (Z - 0)$$

$$\ln\left(\frac{\rho_g(Z)}{\rho_0}\right) = -\frac{\Omega^2}{c_s^2} \frac{Z^2}{2}$$

$$\rho_g(Z) = \rho_0 e^{-\frac{\Omega^2 Z^2}{2c_s^2}}$$

At this point we define H, the *midplane density half-thickness* or *scale height*. $H \equiv \frac{c_s}{\Omega}$ and relabel *Z* and *z* for ease.

$$\rho_g(z) = \rho_0 e^{-\frac{z^2}{2H^2}}$$

This expression tells us how the gas density within the disc varies with height above the midplane.

If we integrate this expression, we get an expression that relates the midplane density half-thickness *H*, to the surface density Σ that we met at the beginning of this section:

$$\rho_0 \int_{-\infty}^{\infty} e^{-z^2/2\,H^2} dz = \sqrt{2\,\pi}\,H\,\rho_0 \equiv \Sigma(r)$$

This quantity can help us to understand the mass distribution within the disc, since it will vary with radius (*r*) from the star. It is useful because this is an observationally resolved value!

<u>Stage 1: Dust settling and coagulation:</u>

We want to know how quickly dust can settle down to the midplane of the disc. To do this, we need to consider the forces that are acting on the dust. The two important forces are the drag force on the dust and the gravitational force on the dust.

Let's start with micron-sized particles. The gas exerts a drag force (which opposes velocity) on the dust grains (Stokes drag) and this causes the dust to drift relative to the gas. Mathematically we write this drag force as:

$$\underline{F}_d \approx -\,a_d^2\,\rho_g\,c_s\,\underline{v}_d$$

Size of dust grain

Velocity of dust relative to gas

Gas density

Sound speed in gas

Let's consider the meaning behind this equation:

- How does dust size influence the drag force? Larger dust grains (big a_d) experience a larger drag force than smaller ones, so they'll slow down faster.
- Dense gas discs will result in larger drag forces. There's more gas to slow down the dust.
- Hotter discs have a higher sound speed. This means that hotter discs have higher drag forces than cooler discs. The gas in a hot disc will have more kinetic energy than in a cooler disc, so this provides more opportunity to exert a drag force on the dust than if the gas was cold.

Q0: Can you do a units check on this equation? Does it make sense?

We can then estimate the acceleration (*a*) of the dust particles and the timescale of this drift (τ_f):

Acceleration is a force per unit mass: $a = \frac{F}{m}$

And we also know that it is a velocity per unit time: $a = \frac{v}{t}$

So, the acceleration of the dust grains must be:

$$a = \frac{F}{m_d} = -\frac{v_d}{\tau_f}$$

(force on the dust divided by its mass, or the dust drift velocity divided by the drift timescale)

The mass of the dust grain could be estimated from its volume (V $\propto a_d^3$) and density (ρ_d):

$$m_d \approx a_d^3 \rho_d$$

So, we can all this use this to estimate the drift timescale (τ_f):

$$\frac{a_d^2\,\rho_g c_s v_d}{m_d} \approx -\frac{v_d}{\tau_f}$$

$$\tau_f \approx \frac{m_d}{a_d^2\,\rho_g c_s} = \frac{a_d^3 \rho_d}{a_d^2\,\rho_g c_s} = \frac{a_d \rho_d}{c_s \rho_g}$$

We can assume that the dust settles vertically into the midplane, and then calculate the timescale for this by assuming our disc is in hydrostatic equilibrium (force balance) and that there is no unpleasant turbulence to make our lives harder!

We balance the gravitational and drag forces:

$$-\frac{v_z}{\tau_f} = g_z$$

And then replace the timescale τ_f with the expression we found above:

$$-\left(\frac{c_s \rho_g}{a_d \rho_d}\right) v_z = g_z$$

We also have an expression for g_z, remember $g_z \approx \Omega^2 z$, and we can rearrange this equation to get the vertical velocity of the dust (v_z).

$$v_z = -\Omega^2 z \frac{a_d \rho_d}{c_s \rho_g}$$

We can use this velocity to get the ***settling timescale***, the time for the dust to travel the distance z at this velocity.

$$t_{settle} = \frac{z}{|v_z|} = \frac{1}{\Omega^2} \frac{c_s \rho_g}{a_d \rho_d}$$

This tells us that big grains settle faster than smaller ones, and that the timescale for settling is smaller for colder discs ($t_{settle} \propto c_s \propto T^{0.5}$).

Q1: Try calculating the settling timescale for a 1mm dust grain at 1 AU, a disc temperature of T = 600 K, a gas density of ρ_g = 10^{-6} kg m^{-3} and a dust density of ρ_g = 10^{0} kg m^{-3}. You may assume the star is a solar mass star, and that the mean molecular weight is $\mu = 0.5$.

We get a settling time of ~**10^6 yr** for 1 mm grains, but it turns out this is too slow! ☹ In order to reduce this settling timescale to **~10^4 yr**, we would need to add more physics into our model. More complete models include:

- Accounting for the fact that grains will collide as they fall and settle. This collisional growth during the settling process makes particles bigger, so they settle faster.
- There will be a spread of velocities of the dust grains, and this means more collisions, so again causes faster settling.
- The grains don't have a perfect spherical shape, they're all nobly. More scientifically, the fractal grain structure increases the drag force and the chance of collisions, decreasing the settling time.

The gas and dust within the disc orbit the star differently.
The orbital acceleration that the gas feels can be described by the following force balance:

$$\frac{v^2}{r} = \frac{GM_*}{r^2} + \frac{1}{\rho}\frac{\partial p}{\partial r}$$

Centrifugal force = gravitational force + gradients in the gas pressure

But the solid dust particles follow Keplerian orbits:

$$\Omega = \frac{2\pi}{P} = \frac{v}{r} = \sqrt{\frac{GM_*}{r^3}}$$

The gas is slightly pressure supported compared to the dust, so the gas orbits more slowly than the solids/planets. The solid particles "feel" a headwind – this causes orbit decay (remember the stages of planet formation in the table above?).

<u>Stage 2: Boulders to planetesimals</u>

Getting from dust to clumps can be explained with our current scientific understanding, but growing these clumps to larger boulders is more of a puzzle. Currently, there's a missing piece of the puzzle, some missing physics that we need to recognise in order to explain the growth of metre-sized bodies to km-sized bodies very *quickly.* There are two possible mechanisms that could be the missing puzzle piece:

1. Gravitational instability – if some bit of the disc gets dense and cool enough to collapse then it could form a planet.
2. Fast Collisional growth – lots of collisions of these bodies leading to them becoming larger bodies. Turbulence might be important here. Turbulence results in instabilities in the disc and therefore denser pockets of dust and gas. This can then lead to growth of this objects. So, turbulence may cause fast collisional growth from pebble sized objects to 100-1000 km-sized objects.

We might not understand how this stage happens, but we know that it must because we know that planets exist!

The next growth stage involves gravitational dynamics. The central star provides the dominant force and the evolution at this stage is driven by gravitational scattering and collisions.

The Hill Sphere is the region around a planet where its gravity is more important that the star's gravity. Specifically, the growing planet's gravity is more important than the tidal force due to the central star.
The expression for the hill radius is:

$$R_H = \left(\frac{m_p}{3M_*}\right)^{1/3} a$$

This is radius is important for determining the planet's "feeding zone", the area around the embryo that can supply more material to the growing embryo. The to-be-planet can only grow from the region where its own gravity dominates, so the dominant embryo planet grows by accreting all the planetesimals within the "feeding zone".

We can estimate the mass of the embryo using the following expression:

$$M = (2\pi a)(2\Delta a)\Sigma_p(a)$$

the distance around the planet's orbit

twice the thickness of the feeding zone i.e. the width of the feeding zone

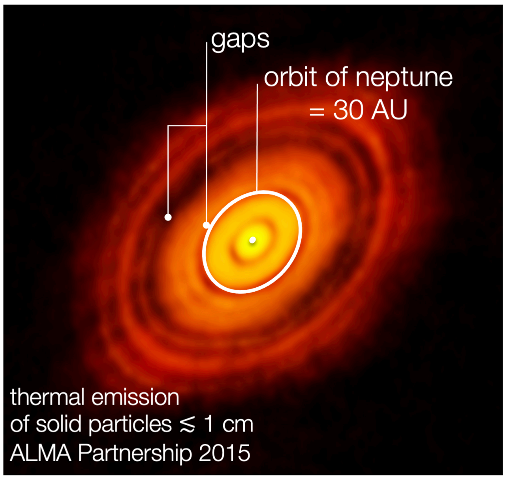
the surface mass density in the z direction of the planetesimals at distance a from the star.

This tells us about the mass within the feeding zone.

The half-width Δa of annulus is given by feeding-zone radius:

$$\Delta a = \tilde{b}\, R_H \text{ , where } \tilde{b} \approx 5 \text{ to } 10$$

(This value for b comes from numerical simulations.) We might think of this feeding-zone radius as the Hill radius, but in reality, the planet embryo is usually not in a perfect spherical orbit and this means it can feed from a volume that's slightly larger than the Hill sphere itself.



The mass of planetesimals in the feeding zone grows more slowly than the embryo mass, so eventually the growing embryo will be starved as no more mass is being supplied to it. The maximum embryo mass is called the ***isolation mass*** and we can calculate it using:

$$M_{\rm iso} = (2\,\pi\,a)(2\,\tilde{b}\,R_H)\Sigma_p(a)$$

$$= 4\,\pi\,a^2\,\tilde{b}\left(\frac{M_{\rm iso}}{3\,M_*}\right)^{1/3}\Sigma_p(a)$$

The image here shows you what the feeding zones might look like. These gaps could be where planets are forming, even though the planet itself can't be seen.

The isolation mass could also be written as:

$$M_{\rm iso} = (4\,\pi\,\tilde{b}\,\Sigma_p)^{3/2}\frac{a^3}{\sqrt{3\,M_*}}$$

The runaway growth of a rocky core stops when the embryo reaches the isolation mass. This is the end of the core-accretion (CA) stage. At this point, further growth can happen by collisional mergers of fewer large bodies (which we call "oligarchs"). An example of this is the Earth and the Moon.

Some estimated formation times are shown below:

- Core of Jupiter: 1-10 Myr – this is good, because remember the gas in the disc is dispersed by 10Myr?
- Terrestrial planets: 100 Myr

These seem reasonable but there are problems with this CA theory e.g. within models we find the Neptune takes too long to form at its current radius. However, all is not lost for CA, there is evidence from orbits of Kuiper-belt objects that Neptune might have migrated outward at some point. So maybe it's ok after all!

At this point, giant planets need to accrete their envelope. The kinetic energy of gas in the outer disc is low (because the gas is cool).

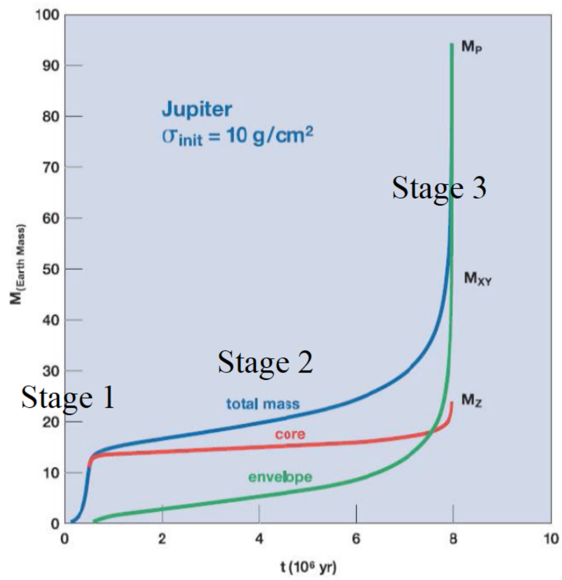
If we start with a core mass **< 10 Earth masses** then:

- $|E_{grav}|$ isn't huge.
- The planetesimal accretion supplies the internal energy (i.e. heats the gas).
- The gas gets heated, so it is only just gravitationally bound. Mathematically this means that the envelope's total energy: $E_{grav} + E_{kin} \sim 0$.
- Our static envelope fills the Hill sphere and so there is no room for more gas to come in and fill it up, and there can be no more accretion!

However, if we start with a core mass **> 10 Earth masses** then:

- There is a deeper gravitational well than before.
- Accretion doesn't supply enough internal energy to heat the gaseous envelope as much i.e. mathematically now: $E_{grav} + E_{kin} < 0$.
- The envelope contracts within the Hill sphere and allows accretion of more gas, so the core can grow further.

We need several Myr to accrete the mass of Jupiter (i.e. about 300 Earth masses). But we know that the gas disc disperses after about 10 Myr. This explains the low gas fraction in Uranus and Neptune. They're further out, there's less gas out there so it's harder for them to accrete.



The standard model for Jupiter formation was published by Pollack in 1996.

- Stage 1: Core grows by planetesimal accretion. This happens pretty quickly.
- Stage 2: Isolation mass reached, slow gas accretion begins.

- Stage 3: Rapid gas accretion, stops when a gap clears in the gas disc (see the next section!).

Answers:
Q0: [N] = [m^2] [kg/m^3] [m/s] [m/s] = [kg m /s^2] = [N]

Q1: first we can calculate the orbital angular velocity at this distance from the star:

$$\Omega = \sqrt{\frac{GM_*}{r^3}} = \sqrt{\frac{6.67*10^{-11}*1.989*10^{30}}{(1*1.49*10^{11})^3}} = 2.002 * 10^{-7}$$

Then, calculate the sound speed in the gas $c_s = \sqrt{\frac{K_B T}{\mu m_H}} = \sqrt{\frac{1.38*10^{-23}*600}{0.5*1.67*10^{-27}}} = 3149$ m/s

$$t_{settle} = \frac{1}{\Omega^2}\frac{c_s\rho_g}{a_d\rho_d} = \frac{1}{2.002*10^{-7}}\frac{3149*10^{-6}}{1*10^{-3}*10^0} = 7.85 \times 10^{13}s = 2.49Myrs$$

Useful resources:
- http://www.khadley.com/courses/Astronomy/ph_205/topics/SolarSystemFormation/index.html#:~:text=In%20the%20core%20accretion%20scenario,the%20disk%20to%20form%20protoplanets
- https://www.youtube.com/watch?v=cnzktuvxAQc

Section 6: Exoplanet migration

Just because a planet forms somewhere, that doesn't mean it will stay there. It was established that Jupiter-like planets form by core accretion outside the snow line, but then came the observations of hot jupiters, including the discovery of 51 Peg b, a Jupiter like planet with an orbital period of 4 days! Immediately the exoplanet community were asking how this could be, and the answer was that it didn't form there but moved there.

Previously in the course we have discussed disc formation, and planet formation (the first two points in the diagram below). This part of the course considers the third step – inward migration of giant planets.

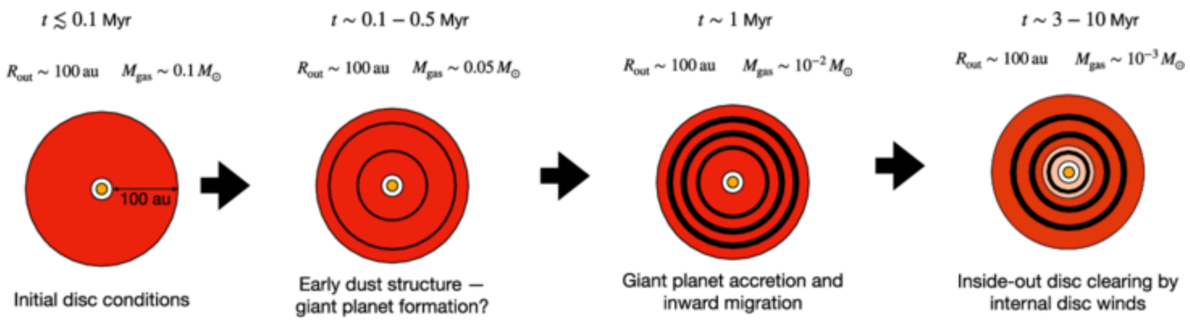


Image credit: Winter+ 2022

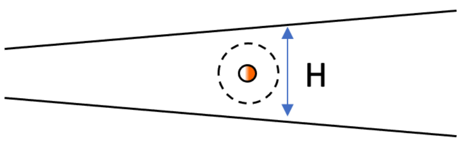
First, we must form our planet inside the gas disc. In doing so, the planet interacts with the gas disc and exerts a torque on gas as it flows by.
There is a region around the planet within which the planet and gas can interact, i.e. a region of interaction, and this is approximately the Hill radius. (Remember that the Hill radius is the radius around the planet where the planet's gravity is dominating over the star's).

$$R_H = \left(\frac{m}{3M}\right)^{1/3} a$$

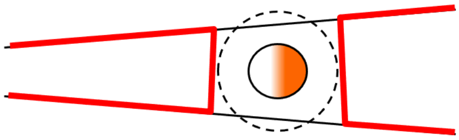
The torque applied to the gas by the planet induces the transport of angular momentum. Remember that a torque is a rate of change of angular momentum?

$$\tau = \frac{dL}{dt}$$

Therefore, if there is a torque then there must be a change in angular momentum. As we will see, the planet loses angular momentum to the gas outside of its orbit. This causes inward planet migration. Planets can cause gaps to open within the disc. This happens if their Hill radius is bigger than the disc scale height.

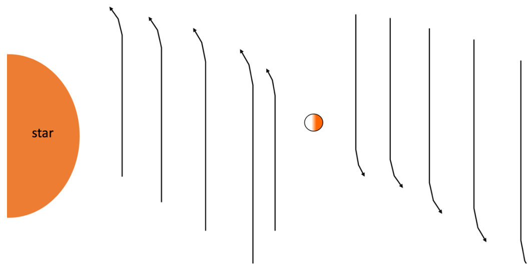


(no gap)

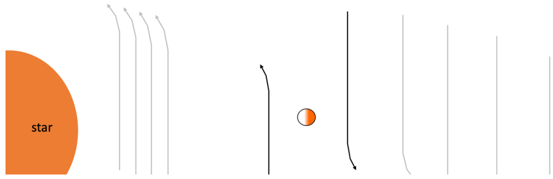


(gap)

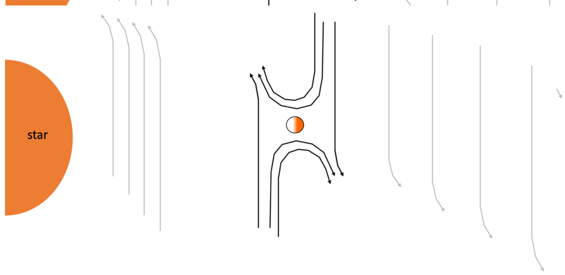
Planet disc interactions:



Let's start with the image on the right and imagine what happens to the gas (black arrows) as the planet and gas orbit the star.



The gas inside the orbit of the planet orbits faster than the planet. Therefore, it loses angular momentum to the planet. This causes the gas to fall inward and orbit faster (image 2).

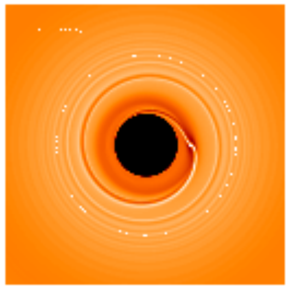


Meanwhile, the gas outside the orbit of the planet orbits slower than the planet. The planet loses angular momentum to the gas, and so the gas moves outward and orbits more slowly (image 2).

But what about the gas very close to the planet? These arrows which represent the net velocity of the flow (or you could think of it as what the planet would see if it was stationary, and the gas whooshed by) should match up. So, we can draw image 3. The gas forms spiral arms that transfer angular momentum from inner disc to outer disc

What does all of this then mean for the planetary orbit? Let's first consider low-mass planets…

Image credit: Frédéric Masse.



Type I

Disc Migration:

Low-mass planets (Type 1 migration):
In the case of a low-mass planet, disc material extends right up to the planet and there is no gap. The planet excites spiral arms, which leads to the inside leading arm and outside trailing arm. Angular momentum is transported throughout the system outwards through the planet.

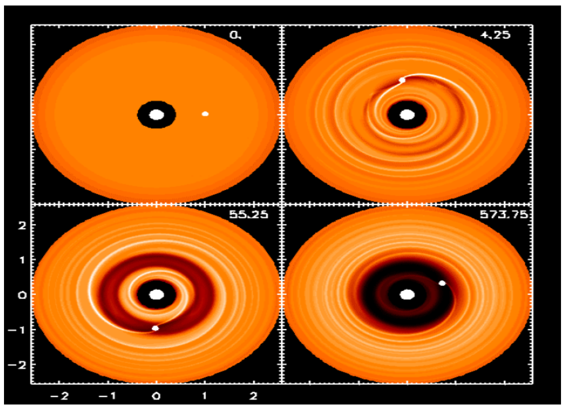
The planet loses angular momentum, and this causes the planet to migrate inwards (usually very rapidly). No gap opens, and this is called ***Type 1 migration***.

It seems intuitive that the migration rate is proportional to the amount of material that "feels" the presence of the planet. More scientifically, this rate depends on the amount of gas inside the Hill sphere.

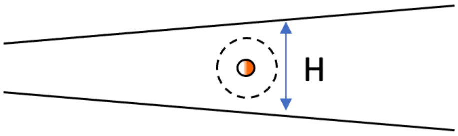
The larger the planet, the faster the migration timescale.

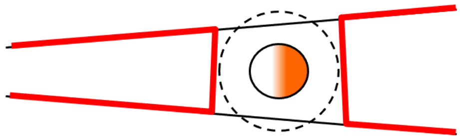
$$t_{\rm migrate} \approx 10^5 \, \frac{0.1 M_{\rm Jup}}{M_{\rm planet}} \text{ years}$$

But more massive cores can open a gap in the disc, an example from (R Nelson, QMUL) is shown in the simulations to the right for a 1 Jupiter-mass planet.

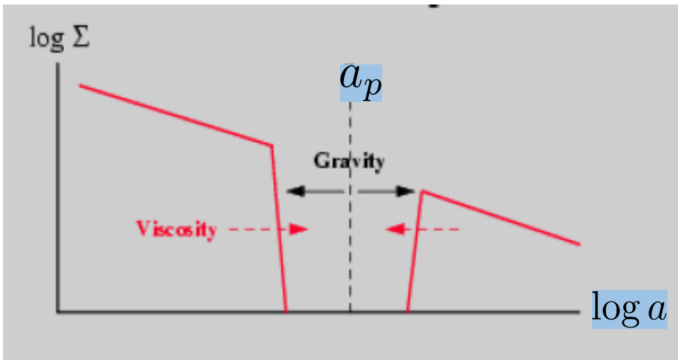


Less massive planets don't clear gap in disc.	More massive planets clear gap in disc
• Low-mass planets • Hill radius < H of disc • Gas flows around Hill radius • **Leads to "Type I" migration**	• High-mass planets • Hill radius > H of disc • Disc truncated at Hill radius from planet • **Leads to "Type II" migration**





Gap width is set by balancing gravity against "viscosity".



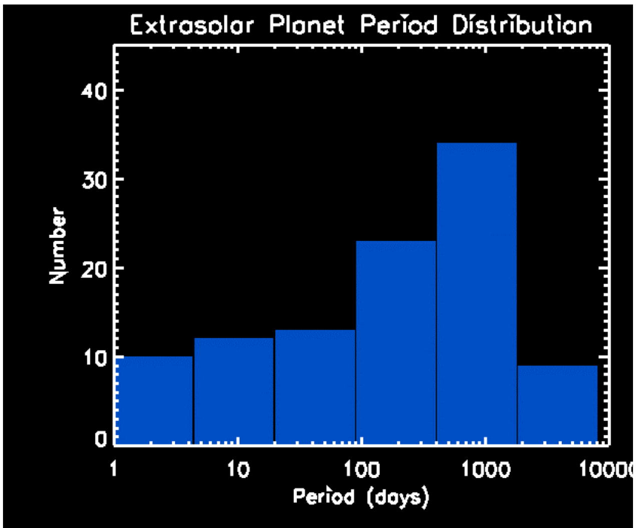
- Gravity causes gap to open through the torques generated.
- But bits of material in the disc is collide with each other.
- They transfer angular momentum so they can move to higher orbits back inside the gap.
- Eventually gravity and viscosity balance and the gap width is set.

<u>Massive planets (type II migration):</u>

This applies to planets more massive than about 0.1 $M_{Jupiter}$ but it does depend on the ratio H/R_H. The gap forms and clears material from near the Hill radius. Viscous disc evolution drives the planet migration and viscosity replenishes gas. The viscous replenishing is the slowest process and therefore, the planet migrates on this viscous timescale:

$$\tau_\nu \simeq \frac{a^2}{\nu} \simeq \text{ few } \times 10^5 \text{ years}$$

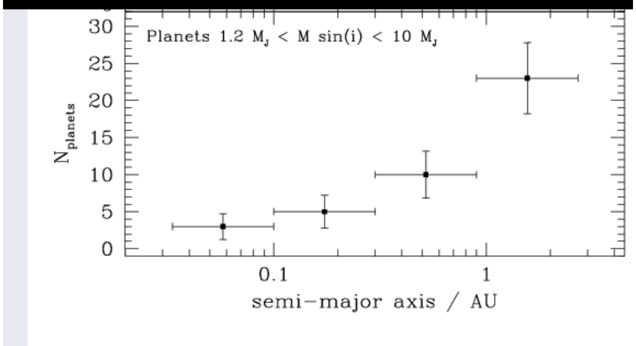
This is about 0.5Myr i.e. < 1Myr. Type II migration can bring Jupiters in from 5 AU in < 1 Myr. This forms hot Jupiters (<1Myrs) while the gas disc is still present (until 10Myrs). But then the obvious question is what stops them? Why don't they migrate inwards until they are swallowed by the star? Why is Jupiter still where it is?



<u>Stalling migration:</u>

Planets form near 5 AU and then migrate inward in a 0.5 Myr timescale. The planet stalls when the disc dissipates, as there is nothing left to allow for angular momentum transport. The probability of finding a planet near the semi-major axis a is:

$$p(a) \propto t_{\text{migrate}} \propto a^2$$

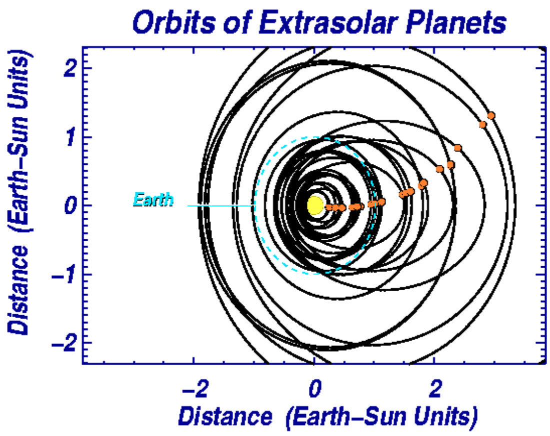


We can plot a distribution of the exoplanet periods and we see a peak around 1000 days.

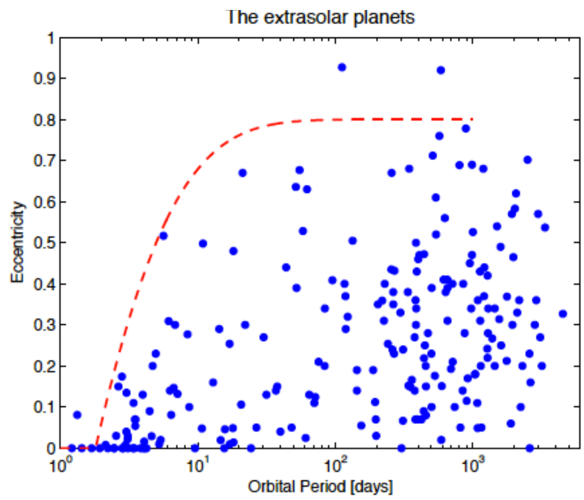
This makes sense because the planet spends more time at large distances from the star, so it is more likely to end up there.

<u>Implications of migration:</u>
We know that migration can form hot Jupiters, they form outside snow-line (a > 4 AU) but then migrate inward to their present location.

However, hot Jupiter migration is inefficient. Many planets merge with the star. Migration will clear planetesimals out of the inner region of the system, leaving no material for terrestrial planet formation… in other words, if Jupiter had migrated in, Earth probably wouldn't be here.



It is logical that migration would affect the planetary orbits. Migration within a disc involves a lot of viscous dissipation and this should lead to circular orbits (the lowest energy state).
Instead, in the observations we can definitely see a population of planets in eccentric orbits.



<u>The scattering of orbits due to planet migration:</u>
If we start with a multiple gas-giant planetary system, which is more likely in metal-rich systems, the planets form relatively close together. The gas keeps the system stable and once the gas is removed, the planets can interact. The evolution of the system then results in chaotic orbits. Orbits can cross, there are close encounters and ejections, and the survivors are left with large eccentricities. The orbits can then recircularise via tidal heating.

Eccentric migration fails if the planet goes inside the Roche limit during eccentric migration. In this case, the planet would be broken apart and fall into the star. Only planets with a(1-e) > a_R (the Roche limit) survive. Tidal forces on the planet lead it to eventually returning to a circular orbit (circularisation) and this process roughly doubles the periastron distance.

Remember this table? We've now covered stage 4:

Stage	Role of gas	Time scale
1. Settling and growth of dust grains	Dust well-coupled to gas	Probably rapid.

2. Pebbles and boulders to km-sized planetesimals	Hydrodynamic drag -> inward motion	Must be rapid.
3. Planetesimals to planet-sized bodies/giant-planet cores	Independent of gas.	Slow.
4. Gas accretion on to giant-planet cores	Orbital migration due to gravitational torques.	Slow.

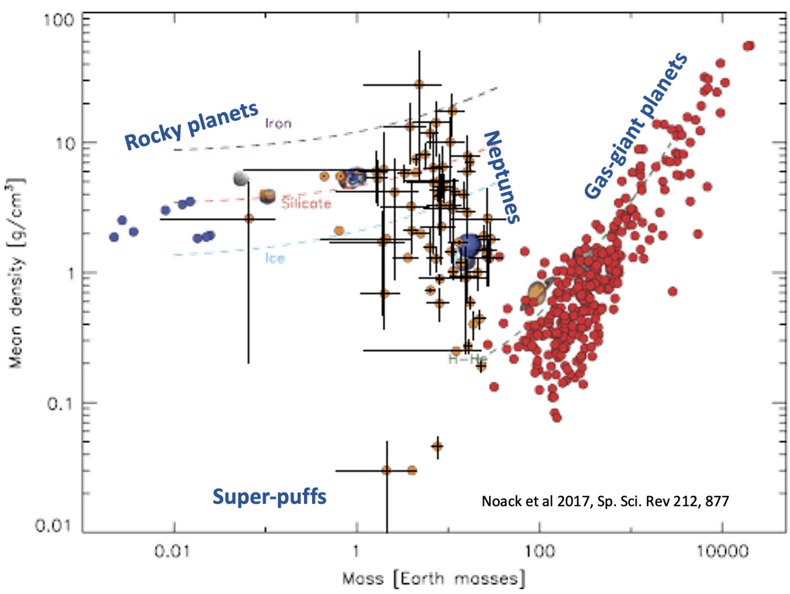
Summary of the migration types we have met:
Type I: disc migration, happens for low mass planets, no disc gap.
Type II: disc migration, happens for high mass planets, disc gap present.
Eccentric migration: caused by planet-planet interaction, eccentric orbit has to circularise (loses energy via tidal heating) and moves in, more likely in metal rich stars (because multiple giant planets).

Useful resources:

- https://www.youtube.com/watch?v=nwSNU3-m0ew
- https://www.youtube.com/watch?v=ko52m9jJGTQ

Section 7: The structure of exoplanets

We know from earlier in the course that we can calculate the planetary bulk density from observational quantities such as the stellar orbital amplitude, the stellar parallax, transit depth and duration, the orbital period and stellar angular diameter.



Remember this?

$$\rho_p = \frac{3K}{a_{\oplus}GP}\frac{\hat{\pi}}{\theta}\left(\frac{R_*}{R_p}\right)^3\left(\frac{a}{R_*}\right)^2$$

Q0: can you remember what all the terms in this expression are?

Ultimately, this expression comes from the following expression for the bulk density:

$$\rho = \frac{M_p}{(4/3)\pi R_p^3}$$

From the plot, we can see multiple branches for different types of planet.

<u>Solid planets:</u>
For solid planets, we use the following three expressions to model their interior. You will have met something very similar in the stellar part of the course, but there will have been an additional equation related to the stellar luminosity. For solid planets, the equations are:

1. Hydrostatic equilibrium: $$\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$$

Our planet is a constant size – it is in **<u>hydrostatic equilibrium</u>**.
i.e. the gravitational force pulling in balances the gas pressure pushing out.

2. Mass conservation: $$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r)$$

The mass of our planet is constant. The mass of a shell between *r* and *r+dr* depends on shell area and local density.

3. An equation of state: $$\rho(r) = F(P(r))$$

We need to know how density depends on local pressure, and for this we need an ***equation of state.*** An equation of state (EOS) is a relationship between ρ and p (and T for gases) and an example that you know well is the ideal gas law. This is the EOS that we use for stars, but for planets it's not so simple. We need something that will work for the solids of our planet i.e. iron, silicates and ice.

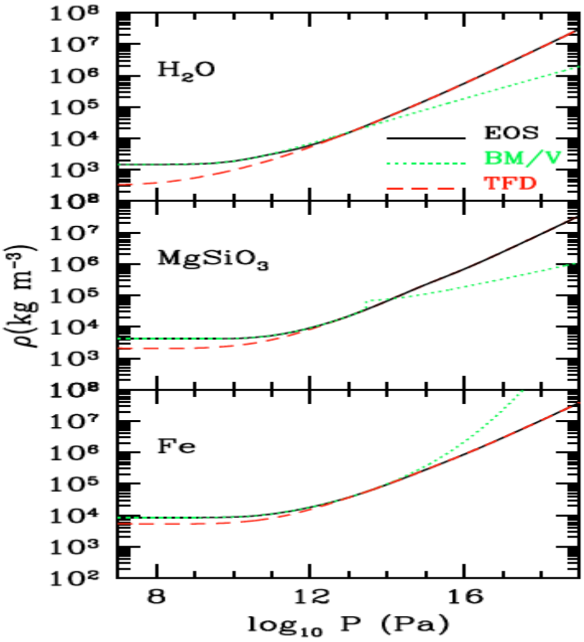
The density of iron, silicates and ice are determined by the Coulomb forces within them. Their density behaviours vary with pressure and pressure in the planet varies with depth. Variation in pressure covers orders of magnitude. We need an approximation that covers the full range of pressures, and it must work for all our substances - water, silicates and iron.

Iron, silicates and ice are incompressible at low pressures (i.e. their densities don't change). In this range, we can determine their behaviour from high-pressure lab experiments. At higher pressures, the chemical bonds break and the atoms are squeeze closer together. Electron degeneracy pressure becomes important – you will learn more about this in later years in quantum mechanics courses!

We can model the equation of state in different ways. In **green is the lab experiments**. Static compression experiments in the lab can reach 2×10^{11} Pa and shock experiments can reach 10^{13} Pa (but very high T).

In **red, the EOS from Thomas-Fermi-Dirac theory** (this is OK at high pressure but not so good for low pressure.) This EOS balances Pauli pressure vs Coulomb forces but it doesn't do chemical bonds (i.e. the lower pressure region).

The **black curve shows the modified polytropic EOS**. This is just an equation that just happens to work well.



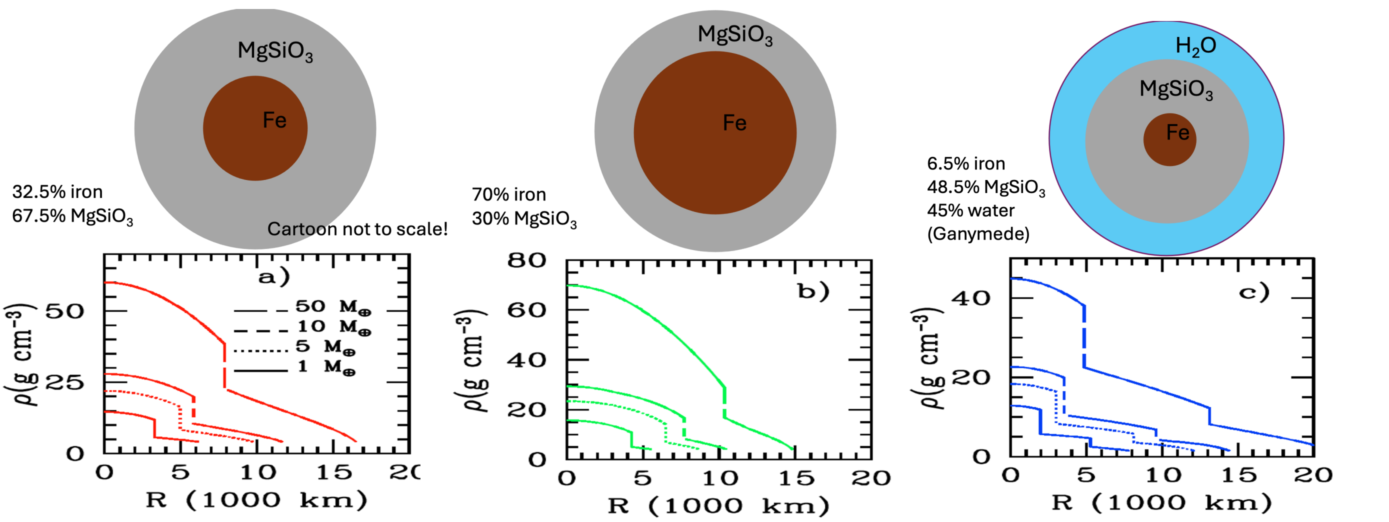
$$\rho(P) = \rho_0 + cP^n$$

We can then find the coefficients (c and n) for different substances and find that as a good approximation, density goes like the square root of pressure.

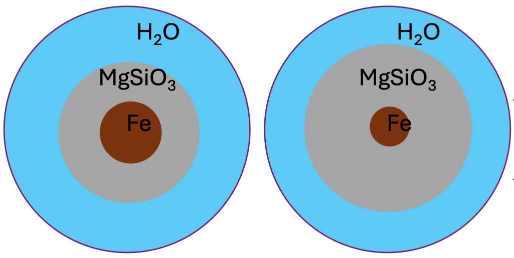
Material	ρ_0 [kg m^{-3}]	c [kg m^{-3} Pa^{-n}]	n
Fe(α)	8300.00	0.00349	0.528
$MgSiO_3$ (perovskite)	4100.00	0.00161	0.541
(Mg,Fe)SiO_3	4260.00	0.00127	0.549
H_2O	1460.00	0.00311	0.513
C (graphite)	2250.00	0.00350	0.514
SiC	3220.00	0.00172	0.537

This expression is only valid for pressures < 10^{16} Pa – but this pressure is about the pressure inside the core of the Sun, so this is ok for planets!

Below are some models showing the bulk density of planets with planetary radius for a range of different planet masses (1, 5, 10 and 50 Earth masses).

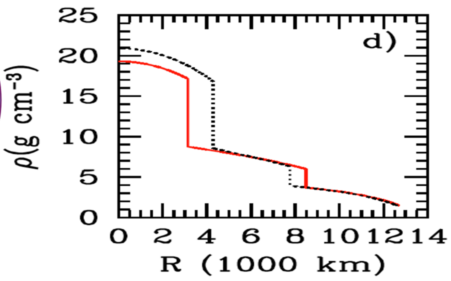


Two planets:
both 6 M_{Earth}, similar radii but
different mix of $Fe/MgSiO_3/H_2O$:



17% iron
33% $MgSiO_3$
50% water

6.5% iron
48.5% $MgSiO_3$
45% water



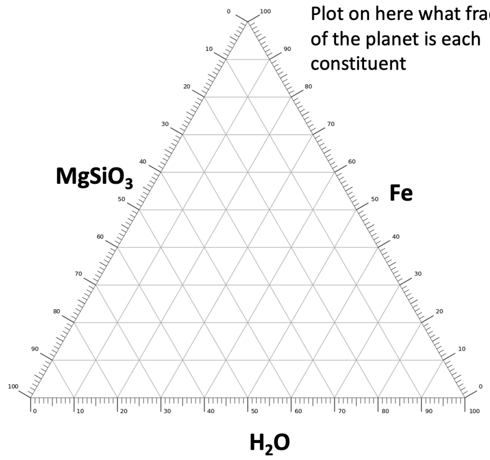
Seager et al 2007, ApJ 669, 1279

The ternary diagram (the Toblerone diagram):

If we think about planets with 3 ingredients, as we have been doing, we can make a **ternary diagram** to describe and compare them.

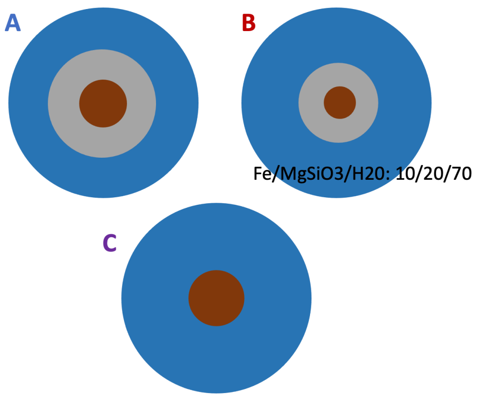
Ternary diagrams are lovely triangular shaped plots as shown on the right.

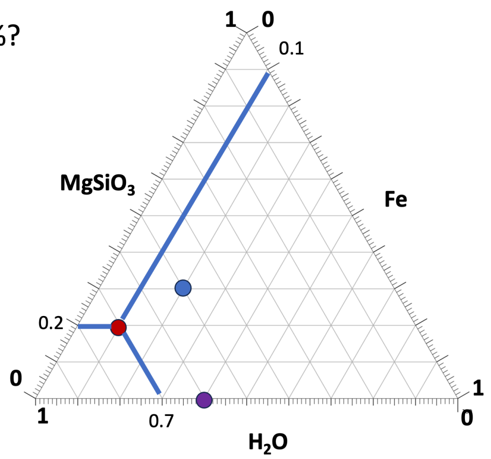
Below is an example showing three planets (with cartoons of their structure). The fraction of each constituent is shown for planet B and the lines showing how to read the plot. Can you find the constituent parts for A and C? The answers are in the lecture slides.



Plot on here what fraction of the planet is each constituent

What are their Fe/MgSiO3/H2O %?





Giant-planet models:

Our modelling for solid planets does not apply to the gas giants. For them we will need an EOS appropriate for gases! In low-density regions, we can use our old friend the ideal gas law.

$$P(r) = \frac{\rho(r)kT(r)}{\mu m_{\rm H}}$$

But deep in the core, the electron wavefunctions may overlap (electron degeneracy again) so we need to adapt this equation. We need to include this electron degeneracy pressure.

$$P(r) = \frac{\rho(r)kT(r)}{\mu m_{\rm H}} + K_{\rm NR}\; n_e^{5/3}$$

Here, n_e is number of electrons per unit volume, and the gas is mainly fully-ionised H and He, so the constant K_{NR} can be written as:

$$K_{\rm NR} = \frac{h^2}{5m_e}\left(\frac{3}{8\pi}\right)^{2/3}$$

(More on this in AS4012, 2 years from now! Don't worry about this, but it is given here for completeness.) Notice that T is important in the ideal-gas law, but not in the degenerate law.

We might want to know what the temperature within the gas giant is. For this, we need to consider that giant planets shrink as they radiate - Jupiter is still contracting and emits about 65% more power in the IR than it receives from the Sun.

<u>Worked example:</u>
Use the Virial Theorem to find interior temperature:

$$|E_{\rm grav}| = 2E_{\rm int}$$

We can use this to make a rough estimate assuming the gas is fully ionised H ($\mu = 0.5$):

$$\frac{GM_{\rm Jup}^2}{2R_{\rm Jup}} \approx \frac{M_{\rm Jup}}{\mu m_{\rm H}} \times \frac{3}{2}kT \Rightarrow T \approx \frac{GM_{\rm Jup}\mu m_{\rm H}}{3kR_{\rm Jup}} = 3.7 \times 10^4 K$$

This ignores the internal density structure but confirms that the gas is hot enough to be ionised so our original assumption was a good one. (But, this temperature is not hot enough for thermonuclear reactions – 1 million K – so we haven't got a star!)

So, the internal temperature is $$T \approx \frac{GM_{\rm Jup}\mu m_{\rm H}}{3kR_{\rm Jup}}$$

This is proportional to planet mass and inversely proportional to planet radius.

- More massive planets with radii ~ 1 R_{Jup} have **hotter** interiors.
- Inflated planets with masses ~ 1 M_{Jup} have **cooler** interiors.
- Jupiter's interior is getting **hotter** as it contracts.

Full models must include the transport equation to describe cooling – just like stars!
Within the convective region:

$$\frac{dT(r)}{dr} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\left(\frac{dP}{dr}\right)$$

Within the radiative region:

$$\frac{dT(r)}{dr} = -\frac{3\kappa\rho(r)}{4acT^3}\frac{L(r)}{4\pi R^2}$$

Section 8: Exoplanetary atmospheres

We are lucky enough to live in a time where we can make observations as well as models of exoplanetary atmospheres! The first thing we might want to understand is what determines if a planet can retain an atmosphere – many exoplanets will start with one, but they might not keep a hold of it for various reasons, e.g. Mercury or Mars.

Some things that determine if the planet could retain an atmosphere are listed below.
Q0: can you think of any others?

- The gravitational strength of the planet - i.e. mass/size will determine the gravitational pull on the atmospheric atoms/molecules.
- The temperature of the atmosphere - e.g. how close the planet is to the star – will determine how much thermal energy the atmospheric particles have to escape the gravitational pull.
- What the atmosphere is made of (e.g hydrogen is much lighter than nitrogen).

Jeans escape as a method of predicting planetary atmosphere retention:

The Jeans escape is calculated from the following method:
Look at the top of the atmosphere and if the following two conditions are met then we say that the species (chemical) in the atmosphere will be lost/escape:

1. if the mean free path of the species > the atmospheric scale height ***and***
2. if the particle velocity > the escape velocity

What do these conditions mean?

1. The distance the particle travels between collisions > distance between particles. This means our particle will keep going without collisions keeping it retained.
2. The particle must be travelling fast enough to escape the planet's gravity. Otherwise, if 1 is met but 2 isn't, the particle will eventually be pulled back down.

We can express this information mathematically, so that we will be able to quantify atmospheric retention:

Calculate the escape velocity by setting the kinetic energy of the molecule equal to the gravitational energy. If these energies are equal, then the particle ***just*** has enough energy to escape:

$$\frac{1}{2}v_{\mathrm{esc}}^2 = \frac{GM}{R} \Rightarrow v_{\mathrm{esc}} = \sqrt{\frac{2GM}{R}}$$

We can calculate the mean kinetic energy of a molecule with mass m using an equation from thermal physics:

$$\left\langle \frac{1}{2}mv^2 \right\rangle = \frac{3}{2}kT$$

This is the mean energy, and you could estimate the mean speed from this by rearranging the equation. To calculate the mean speed "properly" you can use:

$$< v >= \sqrt{\frac{8kT}{\pi m}}$$

This latter expression comes from the peak of the Maxwell-Boltzmann distribution, and it is often called the ***root-mean-square velocity***. Either of these expressions would be acceptable in an exam.

The Jeans escape criterion is **$<v>$ $< v_{esc}/6$** to retain a given species (type) of molecule.

Looking at at these expressions we can see that:

- It is easier to retain heavier molecules e.g nitrogen rather than hydrogen,
- It is easier for planets far out to retain their atmosphere because they have colder atmospheres,
- Is it easier for planets with a larger gravitational acceleration to retain their atmosphere,

Q1: You find an exoplanet with the same radius and temperature as Earth but 2 x the mass. How much easier is it for this planet to retain its atmosphere compared to earth?

Worked example - Jeans escape of planetary atmospheres:

You find an Earth-like exoplanet with radius 0.9 Earth radii, and an equilibrium temperature of 270K. Calculate the escape speed on this planet relative to Earth.

$$\frac{v_{esc}}{v_{esc\ Earth}} = \frac{\sqrt{\frac{2GM}{R}}}{\sqrt{\frac{2GM_{earth}}{R_{earth}}}} = \sqrt{\frac{M}{M_{earth}} \frac{R_{earth}}{R}} = \sqrt{\left(\frac{R}{R_{earth}}\right)^3 \frac{R_{earth}}{R}} = \sqrt{\left(\frac{R}{R_{earth}}\right)^2}$$

$$v_{esc} = \sqrt{(0.9)^2}\ v_{esc\ Earth} = 0.9\ v_{esc\ Earth}$$

Calculate the “root mean square” velocity of oxygen on this planet given that it is 579m/s on Earth. Note that for Earth, Teq = 255K and Vrms = 579m/s.

$$\frac{v_{rms}}{v_{rms\ Earth}} = \frac{constant\ \sqrt{T}}{constant\ \sqrt{T_{earth}}} = \sqrt{\left(\frac{T}{T_{earth}}\right)} = \sqrt{\left(\frac{270}{255}\right)} = 1.03$$

$$v_{rms} = 1.03 v_{rms\ Earth} = 596\text{m/s}$$

Since the escape speed on Earth is 11.2km/s, can this exoplanet retain oxygen?

$$v_{esc} = 0.9 * 11.2\ km/s = 10.08\ km/s = 10080 m/s$$

Need <v> < v_{esc}/6 to retain a given species (type) of molecule and $v_{rms} < 17 v_{esc}$ therefore, oxygen is retained.

<u>Worked example - Jeans escape of GJ1214b:</u>

Can GJ1214b retain hydrogen? It has an equilibrium temperature of 547K, mass of M = 0.0197 M_{Jup} and planetary radius of R = 0.254 R_{Jup}.

Use the Jeans escape criterion:
molecules are retained if: $\langle v \rangle < \frac{1}{6} v_{\mathrm{esc}}$

i.e. $$\sqrt{\frac{8kT}{\pi m}} < \frac{1}{6}\sqrt{\frac{2GM}{R}}$$

Rearrange this to find the minimum mass that can be retained.

$$\begin{aligned} m &> 144\frac{kT_{\mathrm{eql}}R_p}{\pi G M_p} \\ &= 144\frac{1.381\times10^{-23}\times547\times0.254\times6.9911\times10^{7}}{\pi\times6.673\times10^{-11}\times0.0197\times1.898\times10^{27}}\ \mathrm{kg} \\ &= 2.586\times10^{-27}\ \mathrm{kg} \\ &= 1.472 m_{\mathrm{H}}. \end{aligned}$$

Hence minimum molecular mass retained is 1.47 hydrogen atoms, i.e. molecular hydrogen is *just* retained.

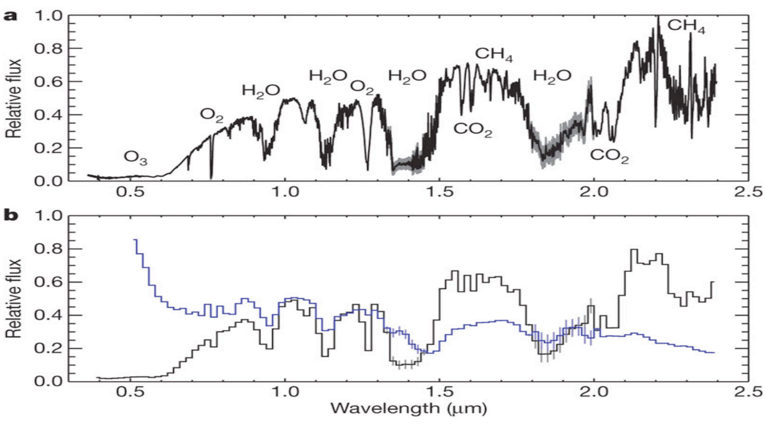
<u>Q2: Jeans escape of planetary atmospheres:</u>

GJ 1132 b is a super Earth exoplanet orbiting an M dwarf. Its mass is 1.66 Earths and radius is 1.13 Earth radii. It has an equilibrium temperature of 529K with an Earth albedo and 409K for a Venus albedo. Calculate the escape velocity and the range of possible rms velocities for molecular hydrogen and determine if it could be retained or not.

Jeans escape is a useful but pretty rough approximation. There are ways of making heavier molecules escape, but you need to provide them with some extra energy in order to do so. Heavier atoms can escape if there are enough collisions with fast-moving, lighter atoms as this provides them with the extra energy needed. This requires a source of energy in the atmosphere and examples of this are planetesimal accretion in young planets or strong irradiation in low-mass hot Jupiters.

An example of this is HD209458. Spectroscopy data shows a 4x deeper transit in the H Lyman-a line than at other wavelengths. This means that is has an extended atmosphere, in fact, it was found to be about the size of the planet's Roche lobe. The atmosphere was being heated by irradiation from the star and the escaping gas ('exosphere') was filling the Roche lobe. **This led to a new technique of measuring transits depths in different wavelengths.**

This was first tested on our own atmosphere during a lunar eclipse! During the lunar eclipse the Moon is in Earth's shadow. Sunlight reaching the Moon has had to pass through Earth's atmosphere. Looking at this reflected moonlight gives a spectrum of sunlight passing through Earth's atmosphere and thus we can see the chemical signatures of the atmospheric gases that have absorbed the sunlight – e.g. water, methane.



This can be applied to exoplanets where it is called "transmission spectroscopy". First, observe a transit with a spectrograph and then measure the transit depth at every wavelength. We can calculate a spectrum ratio, the relative depth of the transit compared to the baseline flux, and we now know that this depends on wavelength:

$$R(\lambda) = F_{\text{transit}}(\lambda) - F_0(\lambda)$$

Both the planet and its atmosphere are causing the transit:
The atmosphere has scale height:

$$H = \frac{c_s^2}{g_p} \approx \frac{kT_{\text{eql}}}{\mu m_{\text{H}} g_p}$$

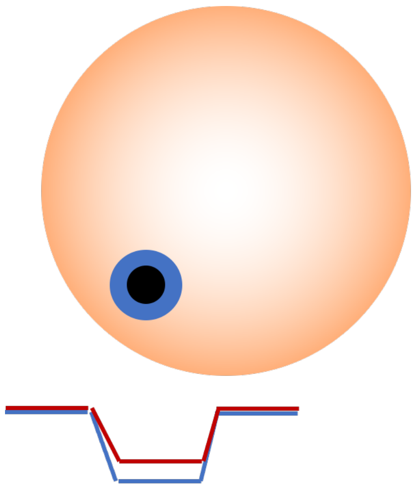
And the area of the planet silhouette responsible for the planetary transit is:

$$A_0 = \pi R_p^2$$

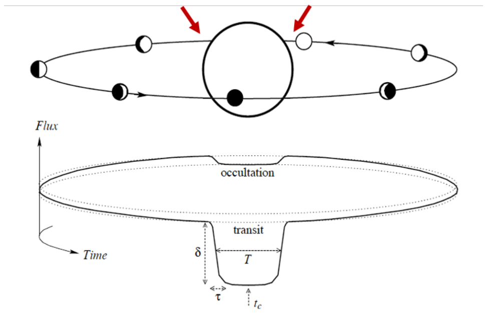
But the area of the planet + atmosphere is responsible for the transit depth in other wavelengths:

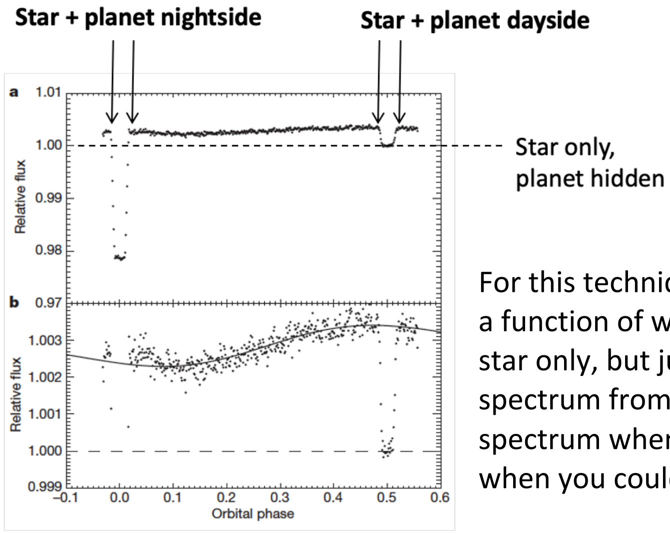
$$A \approx \pi R_p^2 + 2\pi R_p H$$

If the atmosphere is more transparent at some wavelengths than others, the transit depth will depend on wavelength.



Transmission spectroscopy is done during a transit – when the planet is between you and the star. Here we are seeing the nightside of the planet. Another technique is "occultation spectroscopy" – when the planet is going behind the star i.e., we are seeing the spectrum of the dayside of the planet. We can't see the planet as it is *directly* behind the star, but we can see it just before and just after (red arrows on the diagram).





Heather Knutson et al 2007, Nature 447, 183

For this technique, we measure the occultation depth as a function of wavelength. During the eclipse we see the star only, but just out of the eclipse we see the spectrum from the star + planet dayside. Taking the spectrum when you couldn't see planet and subtracting when you could, gives the dayside spectrum!

Answers:
Q0: if the planet has a magnetic field, what kind of star the planet orbits (e.g. how active it is)

Q1: $v_{esc} = \sqrt{\left(\frac{2G(2M_E)}{R_E}\right)} = \sqrt{2}v_{escE}$ and need <v> < v_{esc}/6 and <v> < $\sqrt{2}v_{escE}/6$

=> $\sqrt{2}$ times easier

Q2: $v_{esc} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2(6.67\times10^{-11}\times1.66\times5.97\times10^{24})}{1.13\times6.64*10^{6}}} = 13273\ m/s$

$$v_{rms} = \sqrt{\left(\frac{8K_bT}{\pi m}\right)} = \sqrt{\left(\frac{8\times1.38\times10^{-23}\times529}{\pi\times2\times1.67\times10^{-27}}\right)} = 2359\ m/s$$

$$v_{rms} = \sqrt{\left(\frac{8K_bT}{\pi m}\right)} = \sqrt{\left(\frac{8\times1.38\times10^{-23}\times409}{\pi\times2\times1.67\times10^{-27}}\right)} = 2074\ m/s$$

Need <v> < v_{esc}/6 to retain

With Earth's albedo, $v_{rms} < 5.6\ v_{esc}$ therefore, not retained.

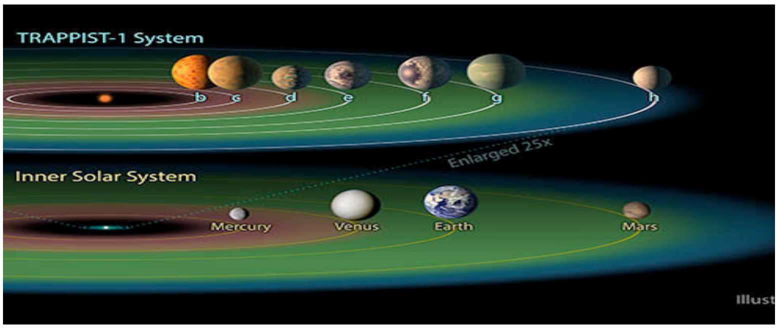
With Venus's albedo, $v_{rms} < 6.4\ v_{esc}$ therefore, retained.

GJ1132b ***may*** retain molecular hydrogen.

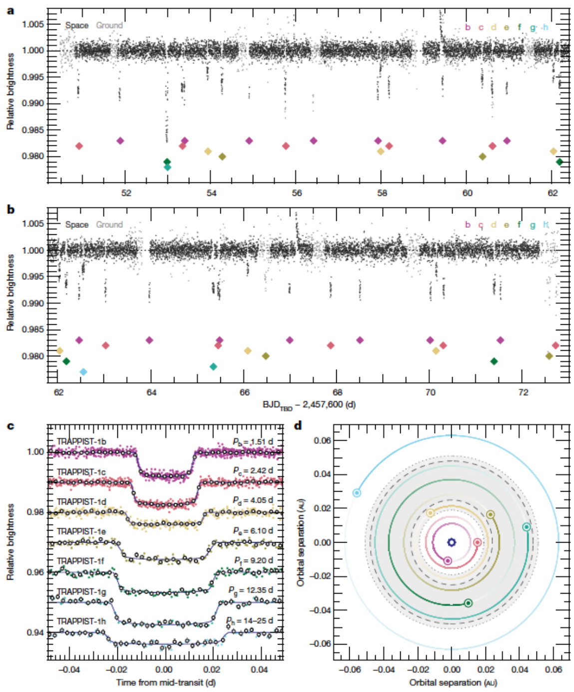
Section 9: Aliens? Exoplanet habitability

The desire to find exoplanets is partly driven by our need to contextualise our own Solar System and Earth. We want to know how common a planet like our own is. It's clearly special to us, but is it common? Specifically, is there any life out there? This drives the search for habitable exoplanets which is a growing area that is, unsurprisingly, very popular amongst the public and funding bodies! The search to find life within our own Solar System continues, but with growing numbers of exoplanets discovered every year, the hunt is now on around other stars too. A new age has begun with JWST, which is looking at atmospheres for hints of "building blocks of life".

Below is a famous system, Trappist-1.



The star has a mass 8% that of the Sun and there are 7 planets in the system. 3 of the planets are Earth sized and the planetary equilibrium temperatures allow for liquid water on their surfaces! To the right you can see their transit curves!



We might start by asking, what does a planet need to be habitable to us? This is clearly a complex question and there are many requirements, a few are listed below:

- Liquid water,
- A magnetic field - to protect the planet from cosmic rays, atmosphere retention, protect from stellar activity,
- An oxygen rich atmosphere,
- A star that isn't too active,
- The planet can't have too large a gravitational pull (we'd get crushed),
- A solid planet,
- A stable ecosystem,
- Maybe a large moon! Suggestions that our Moon could be very important RE season stability and tides.
- … can you think of any others?

The previous question (whilst clearly difficult to answer because of many interconnecting requirements) is it turns out, one of the easier questions to answer. What about when we hunt for exoplanets and ask if they could be habitable?
The next question is "habitable to who?" We would be happy to find any life in the Universe (probably!) and we aren't exclusively looking for life like us. So, in this sense we could mean ***any life*** (not necessarily complex). It then immediately follows to ask "should all life be like us?" The answer is probably no, but we only have one data point to work with – life on Earth…

So, we can ask "what does a planet need to be habitable?" and the answers might go something like this:

- Liquid water… maybe? It seems vital to life on Earth, but perhaps some other chemical could provide this role for life elsewhere.
- A magnetic field to protect the planet from cosmic rays, atmosphere retention, protect from stellar activity. This one seems likely, retaining an atmosphere (whatever that may be) and protecting against X-rays and cosmic rays seems like it would be sensible regardless of the life.
- An oxygen rich atmosphere… maybe? Again, perhaps another chemical could replace the role of oxygen on our planet.
- A star that isn't too active. This again seems sensible – CMEs are bad for all atmospheres and Xray flares will probably be bad for all life.
- Planet can't have too large a gravitational pull – if we want complex life, then presumably there will be some limit to the gravitational strength of the planet that would be sensible for the life to move around. This might not be the case for microbes though…
- A solid planet…probably a gas giant is no good but again, who knows, maybe there is an exception for some microbes?
- Star should live long enough for life to develop. Life will take time to develop (about 4 billion years on Earth – fossil records go back to 3.7billion) so the star needs to live long enough for this to happen.

Habitability indicators are properties of a system we could look at to determine if it might be habitable or not, examples are:

- Orbital period of the planet,
- Bulk composition of the planet,
- Planetary atmosphere presence and composition,
- The metallicity of the star – higher metallicity – longer period of habitability (Danchi and Lopez, APJ 769:27 (16pp), 2013, if you want to know more ☺)
- The type of star – lifetime determines if life can develop, "Habitable Zone (HZ)"…

When talking about the habitability of a planet/moon we soon hit upon a problem (as we have here):
We aren't asking a very well-defined question with **"what makes a planet habitable?"** this could mean a lot of things, habitable for what kind of life? And if not Earth-like life, then what? What would this look like, and therefore what environment would it need? And there is the additional problem - we don't really know how to define it better, because we only know about the kind of life that exists on Earth.

So, let's assume we're looking for life that is **similar to Earth-life**. This restricts our search, but at least we know what we're looking for (kind of)…

<u>One definition of habitability:</u>
The obvious first question when looking for habitability for Earth-like life is: can the planet support liquid water? A more scientific way of asking this question is to ask ***"is the planetary equilibrium temperature consistent with the existence of liquid water?"***

The **<u>habitable zone</u> (HZ)** is the range of orbits that a planet could take around the host star and be able to support liquid water on the planetary surface.

If a planet with Bond albedo *A* radiates as a blackbody of temperature T_P, then:

$$4\pi R_p^2 \sigma T_p^4 = \frac{L_*}{4\pi a^2} \pi R_p^2 (1 - A)$$

we can scale this to Earth and assume that our exo-Earths have the same greenhouse effect, therefore the same Bond Albedo (see later).

$$\Rightarrow \frac{T_p^4}{T_\oplus^4} = \frac{L_*}{L_\odot} \left(\frac{a}{a_\oplus}\right)^{-2}$$

<u>Remember:</u>
The ***bond albedo*** tells us about the fraction of the power that gets reradiated into space. 1 = reflects everything back into space and 0 = reflects nothing.

<u>Keeping a comfortable distance:</u>
So, how close can we get to the star and be in the HZ/ support liquid water? Use the equation from above with the main-sequence mass-luminosity relation:

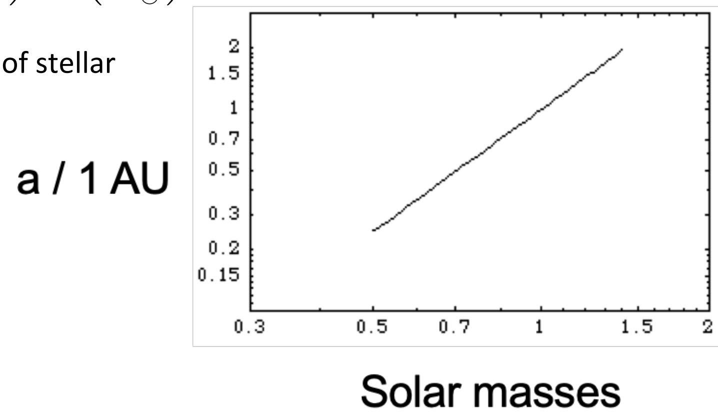
$$\Rightarrow \frac{T_p^4}{T_\oplus^4} = \frac{L_*}{L_\odot} \left(\frac{a}{a_\oplus}\right)^{-2}$$

$$\frac{L_*}{L_\odot} \approx \left(\frac{M_*}{M_\odot}\right)^4$$

If we then fix the planetary temperature at Earth value, we can get a relation for the HZ with stellar mass:

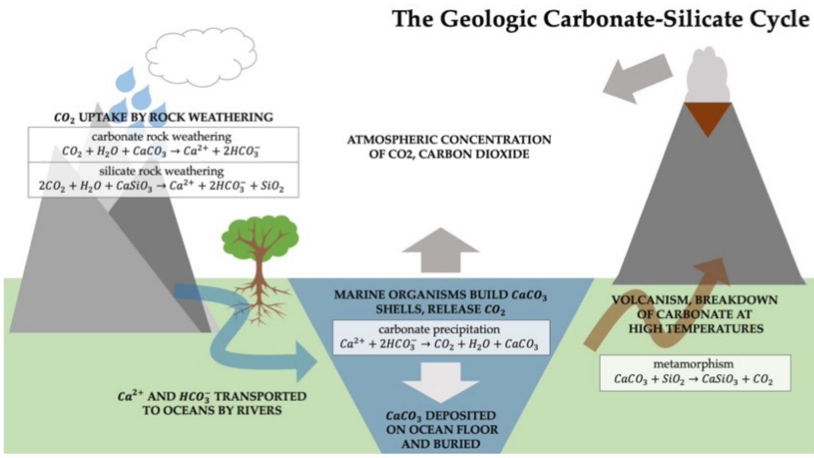
$$\Rightarrow \left(\frac{a}{a_{\oplus}}\right) = \left(\frac{M_*}{M_\odot}\right)^2$$

Once we have determined *a* as a function of stellar mass, we can plot it!

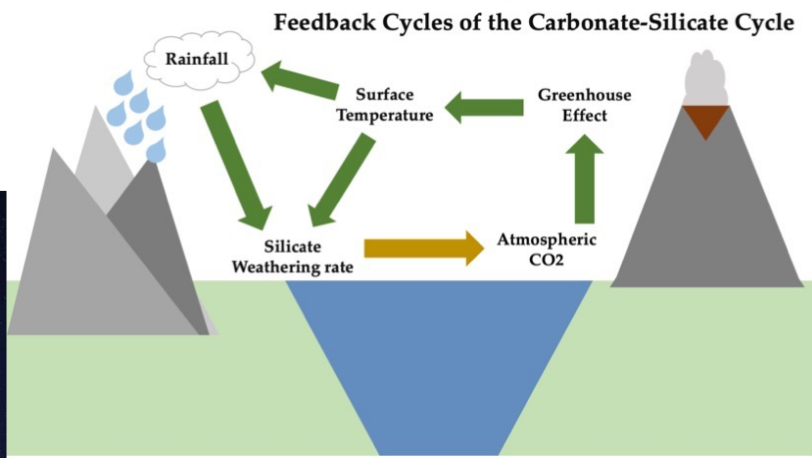


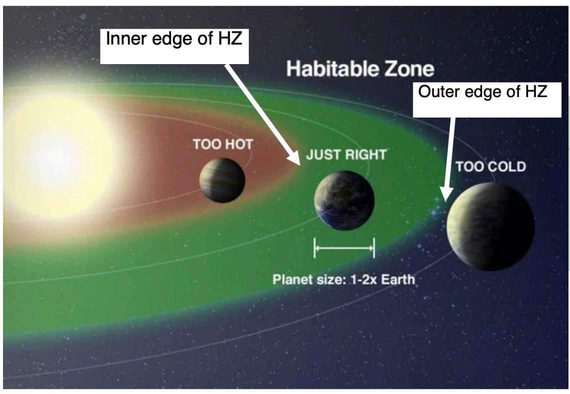
Habitability requires more than this, but thinking about planets where liquid water can be supported on the surface is a good start. We can extend this idea of the habitability zone in various ways. A common HZ definition involves taking H_2O and CO_2 as the main greenhouse gases and involves the carbonate-silicate cycle. This cycle controls the CO_2 levels over long timescales.

- ➔ CO_2 from volcanos enters atmosphere
- ➔ falls with rain
- ➔ weathers rocks
- ➔ transported to rivers and ocean
- ➔ deposited and buried
- ➔ back to volcanos.



This cycle is important with respect to the "edges" of the habitable zone because the CO_2 and H_2O act as greenhouse gases to heat the planet.





In this definition of the habitability zone, the edges of the HZ get a bonus from the CO_2 greenhouse effects that help to keep the planet warm.

Just within the outer edge of the HZ:

- It is almost too cold for liquid water.
- If the oceans freeze:
  - silicate weathering stops but
  - volcanism continues.
  - CO_2 then builds up in atmosphere and
  - The planet warms.

The planet thus manages to keep its liquid water.

Just beyond the outer edge of the HZ:

- The oceans freeze.
- silicate weathering stops but
- volcanism continues.
- CO_2 clouds form as before, but now the many clouds act against us.
- Being covered in clouds leads to an increase in the bond albedo of the planet.
- It reflects more light (and heat),
- so the planet stays cold.

$$4\pi R_p^2 \sigma T_p^4 = \frac{L_*}{4\pi a^2} \pi R_p^2 (1 - A)$$

The inner edge of the habitability zone:

As we move closer to the star, the planetary temperature (T_P) rises. Thus, more H_2O evaporates.

At solar flux $f = 1.1\, f_{\text{here,now}}$ the stratosphere becomes wet.

- Water therefore diffuses into the upper atmosphere,
- UV light dissociates the water molecules,
- the water is lost to space (bad).

At $f = 1.4\, f_{\text{here,now}}$, the oceans evaporate entirely…

- Runaway greenhouse (very very nasty… poor Venus.)

Our current day Solar System:

$$f_{\text{inner}} = \frac{f_{\text{here, now}}}{0.95^2} = 1.1 f_{\text{here, now}}$$

The present inner edge is 0.95 AU and beyond this, loss of water via photolysis, and hydrogen escape.
The present outer edge is 1.37 AU and beyond this, the formation of CO_2 clouds.

$$f_{\text{outer}} = \frac{f_{\text{here, now}}}{1.37^2} = 0.53 f_{\text{here, now}}$$

This assumes a HZ with a carbonate-silicate thermostat, and requires:

- Water
- Subduction of oceanic crust
- Volcanism

But stars are not constant throughout their life, their luminosity (amongst other things) will change and thus their HZ changes too. Really, we want somewhere that is continuously habitable, or else we are in trouble!

Suppose we want the carbonate-silicate thermostat to operate continuously from ~700 Myr to 5 Gyr.

Inner limit of HZ:

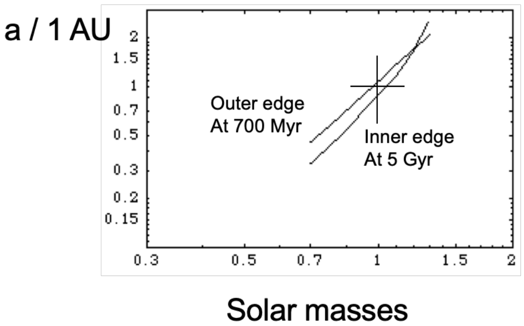
- No loss of water before star is 5 Gyr old.

Outer limit of HZ:

- No CO_2 clouds when star is only 700 Myr old.

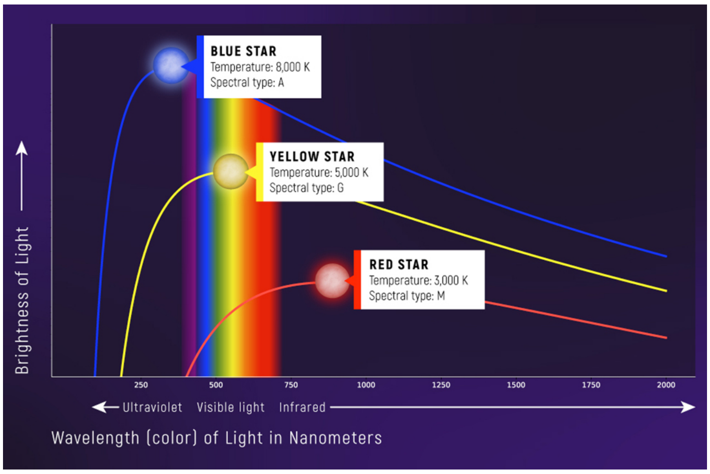
$$\frac{a_{\text{inner}}}{a_{\oplus}} = 0.95\left(\frac{L_{*,\ 5\text{Gyr}}}{L_{\odot}}\right)^{1/2}$$

$$\frac{a_{\text{outer}}}{a_{\oplus}} = 1.34\left(\frac{L_{*,\ 0.7\text{Gyr}}}{L_{\odot}}\right)^{1/2}$$



<u>Stellar spectral type and the planetary albedo:</u>

The spectral type of a star determines the peak of the stellar spectrum and therefore influences the planetary albedo.



<u>Going from G type towards A type:</u>

A higher proportion of incident starlight is Rayleigh scattered if the spectral peak is shifted to blue.

The albedo of planet is therefore increased (more light is reflected).

The inner edge of HZ moves inward to accommodate for this.

<u>Going from G type towards M type:</u>

A higher proportion of incident starlight is absorbed by H_2O, CO_2 if the spectral peak is shifted to red.

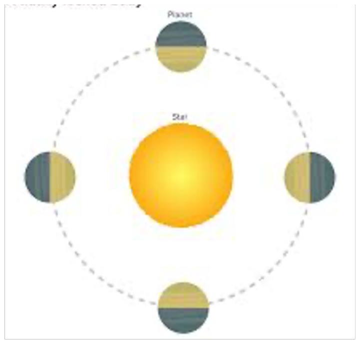
The albedo of the planet is therefore decreased (more light is being absorbed)

The inner edge of HZ moves outward to accommodate for this.

Where to live long and prosper

The Sun is near the upper mass limit for continuous habitability.

G and K stars seem like secure places to live, but what about M dwarfs? We already know that the peak of their spectral energy distribution means the HZ will have to be closer to the star. It doesn't take long to realise that this isn't ideal – impacts from coronal mass ejections (CMEs) and X-ray flare irradiation will be more likely in this case. Do M dwarfs have any other problems? The answer is yes, as we're about to discuss, but they do have the advantage that there are just so many more of them than other stars. They have a statistical advantage of hosting life in this sense, but not in others… Let's talk below about what we should do in order to live long and prosper on our exoplanet.



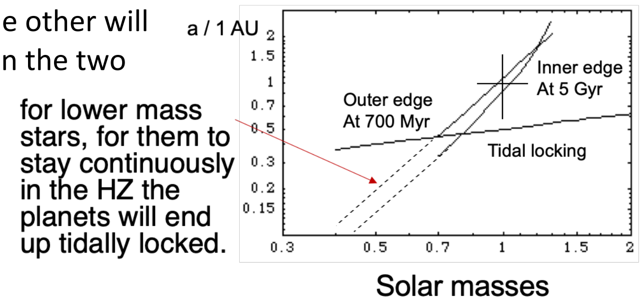
1. Avoid tidal locking

Rotation of an Earth-like planet becomes tidally locked after 4.5 Gyr at orbital distance

$$\frac{a_{\mathrm{lock}}}{a_{\oplus}} \approx 0.5 \left(\frac{M_*}{M_\odot}\right)^{1/3}$$

(Goldreich & Soter 1966; Peale 1977; Kasting et al 1993; Rasio et al 1996; Joshi et al 1997)

If we find ourselves tidally locked then there will be no proper seasons. One side of the planet will be boiling and the other will be freezing. This will cause extreme weather between the two sides, because of the temperature and pressure differences. This is pretty hostile to life… unfortunately for our low-mass star friends, this is far more of a problem around them because our continuously habitable zone will also be within the tidally locked region.



2. Get a magnetic field!

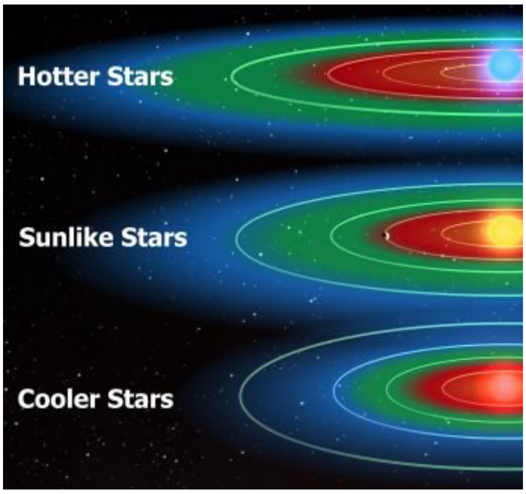
A magnetic field helps to protect the planet from stellar activity like coronal mass ejections – which is both important directly for life and for atmosphere retention (indirectly for life!). It also helps to retain the atmosphere by providing an additional force to any ions.

3. Find the right star

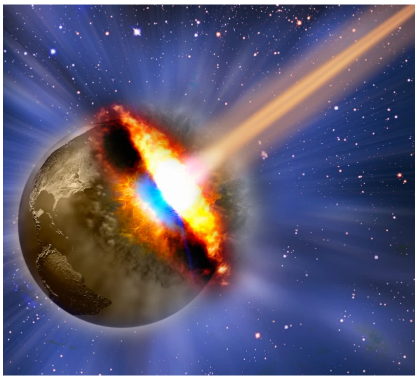
Below are some conditions to help us find the right star:

- Massive stars probably don't live long enough for life to develop on their planets – so we can immediately discount them, although we already know they also have other problems.
- The habitable zone will be closer in for M dwarfs than K or G stars, so the planet is more likely to be bombarded by CMEs, X-ray flares etc.

- CME bombardment can ablate and strip atmospheres, so a star that is very active will be problematic.
- X-rays are bad for life. Low mass stars make more X-rays when they're younger than when they're older. M and K type might have higher X-ray output for longer into Main Sequence than G-type stars (i.e. mass is important too).
- But younger stars live longer… and there are many more of them…



4. **<u>Have the right neighbours</u>**



We want to avoid lots of impacts…some are inevitable but more of them (and with higher velocities) will greatly increase the chance of wiping out all life. The perihelion velocity of comets (which could then cause impacts) can be calculated:

$$\frac{v^2}{v_\oplus^2} = \left(\frac{M_*}{M_\odot}\right)\left(\frac{a}{a_\oplus}\right)^{-1}$$

The HZ radius varies with stellar mass:

$$\left(\frac{a}{a_\oplus}\right) = \left(\frac{M_*}{M_\odot}\right)^2$$

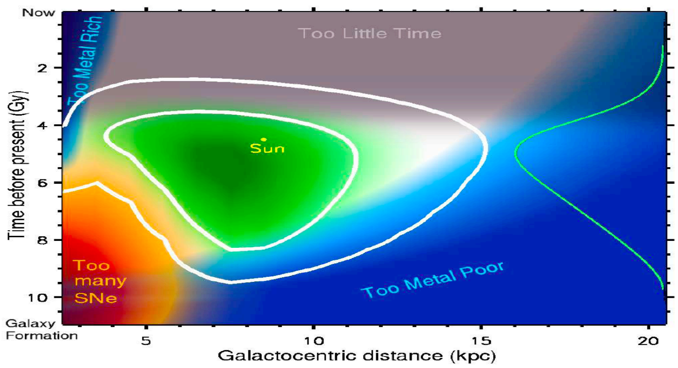
So, the impact energy (related to v^2, think of the kinetic energy) in the HZ scales inversely with mass.

$$\frac{v^2}{v_\oplus^2} = \left(\frac{M_*}{M_\odot}\right)^{-1}$$

i.e., a low mass star is worse in this case.

5. **<u>Avoid other natural disasters…</u>**

I'm sure you could think up other natural disasters that we might wish to avoid, another example of this would include being too close to a supernova – which would immediately wipe out life. We don't have control of when a star would go SN, but we do know that there are some regions of the galaxy where there are lots of stars that will go supernova nearby. This means there are some locations in the galaxy that are not prime real estate…



Some questions for you:

1. How does stellar age influence planetary habitability?
2. How does stellar mass influence planetary habitability?

Useful resources:
* https://iopscience.iop.org/article/10.1088/0004-637X/769/1/27/pdf#:~:text=Stars%20having%20significantly%20higher%2Dthan,than%20do%20low%2Dmetallicity%20stars.