Fluids Notes

PH4031

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Lecture 1

Topics in the course:
This course is an introduction to fluid mechanics. Through these lectures, we will cover topics such as:

- What is the essential physics of fluids?
- What is vorticity?
- Why do boundary layers form?
- How do sound waves travel?
- And what happens when they form a shock?
- What happens when fluids are unstable?

Introduction
What is a fluid? A fluid is simply something that flows. It does not need to be a liquid – it could be a solid or gaseous. Dust or stars colliding may be modelled by fluid equations, and both neutral gases and plasma (ionised gas) can be too.

Towards the end of the course, we will talk about formation of waves on surfaces. This is related to surface tension which owes its behaviour to viscosity.

Fluid equations
If something can be described as a fluid, we can apply "the fluid equations" to mathematically describe it. The fluid equations are a set of equations that are based on the theoretical concept of a "fluid element" – a patch of the fluid over which we define local variables (e.g. ρ, T etc). In order to apply the fluid equations, the following conditions must be met: the size of the patch is such that:

$$(1)\ L_{el} \ll L_{scale} \approx \frac{q}{|\nabla q|}$$

L_{el}: length of the fluid element
L_{scale}: length scale over which q varies by the order of unity.
q: any quantity

Equation (1) means that the fluid element must be small enough that we can ignore systematic variations of quantities across it. i.e., over a fluid element there must not be any variation in property e.g. T.

$$(2)\ n\, {L_{el}}^3 \gg 1$$

n: number density (m^{-3})
Equation (2) requires that the fluid element be large enough that it contains sufficient particles to ignore fluctuations due to a finite number of particles. i.e., there needs to be a reasonable number of particles so that statistical representation is fair.

$$(3)\ L_{el} \gg \lambda$$

λ: mean free path
Equation (3) requires that the fluid element is large enough that constituent particles "know" about local conditions through colliding with each other. This is only needed for collisional fluids (which everything in this course will be!).

NOTE:
1. Fluid elements are just conceptual quantities.
2. L_{el} doesn't enter the fluid equations BUT conditions (1)-(3) limit the applicability of the equations.

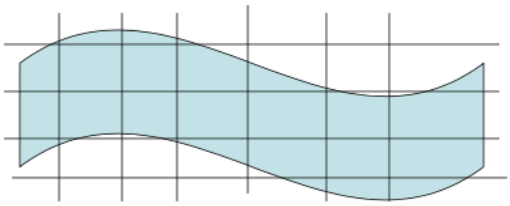
A collisional fluid at a given temperature (T) and density (ρ) will have a well-defined distribution of particle speeds in the local rest frame and hence, it has a corresponding pressure. Thus, one can derive an equation of state for a collisional fluid that fulfils conditions (1)-(3).

<u>Choosing the best description - Eulerian vs Lagrangian methods</u>

For a non-equilibrium situation, if we wish to measure temperature as a function of position and time, we can choose from two methods: Eulerian and Lagrangian. Let's consider them by imagining we have a river and wish to measure the temperature variation in the water with time.

(1) <u>Eulerian Method ("grid method"):</u>

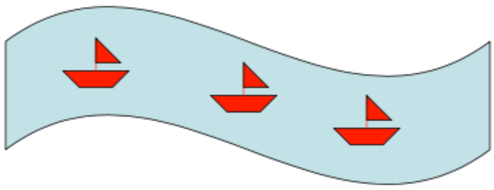
What we need to do:
- Set up a grid: **<u>r</u>** = (x, y)
- Place thermometer at each point in the grid, T(x, y)
- Read thermometers over time T(**<u>r</u>**, t)



In this method:
- Independent variables (**<u>r</u>**, t)
- Variation with time is partial derivative: $\partial T / \partial t$
- Evaluated at a fixed position: **<u>r</u>**

(2) <u>Lagrangian Method ("boats method" or co-moving method):</u>

What we need to do:
- Many boats, labelled "a" each with a thermometer, T(**<u>a</u>**, t)
- Release them in a river, recover them later.



In this method:
- Independent variables (**<u>a</u>**, t)
- Position now depends on time: **<u>r</u>** = r(**<u>a</u>**, t)
- Variation with time is a full derivative: dT / dt

<u>Summary</u>

Eulerian:
- The world is viewed from a fixed position, $\partial Q / \partial t$
- **<u>r</u>** and t are independent

Lagrangian:
- The world is viewed from comoving with fluid, dQ / dt

- r depends on t

Advantages and Disadvantages

Eulerian: Good for steady state: ${}^{dQ}/_{dt} = 0$

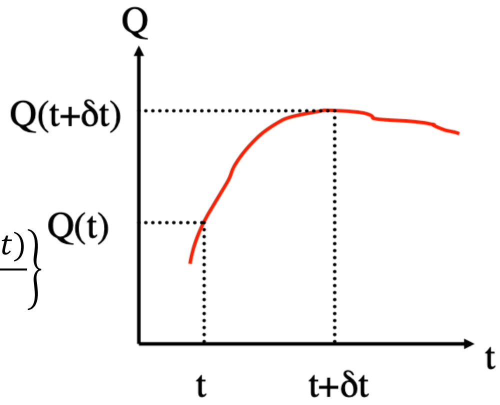
Lagrangian: useful when behaviour of individual elements matter (usually don't)

Lecture 2

Relating Eulerian and Lagrangian descriptions:

Let's start with a mathematical reminder! Consider a function Q(t), the definition of differentiation is:

$$\frac{dQ}{dt} = \lim_{\delta t \to 0} \left\{ \frac{Q(t+\delta t) - Q(t)}{\delta t} \right\}$$

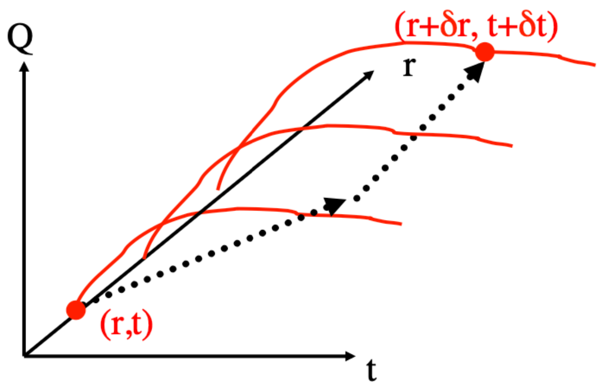


Now if we take Q as a function of two variables we find:

$$\frac{dQ}{dt} = \lim_{\delta t \to 0} \left\{ \frac{Q(\boldsymbol{r}+\delta \boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t)}{\delta t} \right\}$$

$\boldsymbol{r}$: position of a fluid element at a time, t

$\boldsymbol{r} + \delta \boldsymbol{r}$: position of a fluid element at time $t + \delta t$



This feels difficult to evaluate because we can only take variations one at a time, so we should split them up:

Looking at the numerator of $\frac{dQ}{dt}$:

$$\begin{aligned} Q(\boldsymbol{r}+\delta\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t) &= Q(\boldsymbol{r}+\delta\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t) + Q(\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t+\delta t) \\ &= Q(\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t) + Q(\boldsymbol{r}+\delta\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t+\delta t) \end{aligned}$$

The first term shows the variation in time at a fixed r and the second gives the variation in r at fixed $t + \delta t$. In other words, we can deal the variations in position and time separately.

- $Q(t+\delta t) - Q(t)$ at fixed $\boldsymbol{r}$
- $Q(\boldsymbol{r}+\delta\boldsymbol{r}) - Q(\boldsymbol{r})$ at fixed $t + \delta t$

Remember the Taylor expansion:

$$f(x) = f(a) + (x-a)f'(a) + \cdots$$

We are going to use this to expand our functions $Q(t+\delta t)$ and $Q(\boldsymbol{r}+\delta\boldsymbol{r})$.

We can then write the numerator as an expansion in $\delta\boldsymbol{r}$ and δt, remember:

$$Q(t+\delta t) = Q(t) + \delta t \frac{\partial Q}{\partial t} + \cdots$$

evaluated at t!

$$Q(\boldsymbol{r}+\delta\boldsymbol{r}) = Q(\boldsymbol{r}) + \delta\boldsymbol{r}.\boldsymbol{\nabla} Q + \cdots$$

evaluated at $t+\delta t$!

and by truncating and taking only the linear terms:

$$Q(t+\delta t) - Q(t) \approx \delta t \frac{\partial Q}{\partial t}$$

$$Q(\boldsymbol{r}+\delta\boldsymbol{r}) - Q(\boldsymbol{r}) \approx \delta\boldsymbol{r}.\boldsymbol{\nabla} Q$$

And adding these two equations gives the numerator of the original equation:

$$Q(\boldsymbol{r}+\delta\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t) \approx \delta t \frac{\partial Q}{\partial t} + \delta\boldsymbol{r}.\boldsymbol{\nabla} Q|_{t+\delta t}$$

$$\approx \delta t \frac{\partial Q}{\partial t} + \delta\boldsymbol{r}.\left\{\boldsymbol{\nabla} Q + \delta t \frac{\partial(\boldsymbol{\nabla} Q)}{\partial t}\right\} *$$

Bringing everything together:

$$\frac{dQ}{dt} = \lim_{\delta t \to 0} \left\{\frac{Q(\boldsymbol{r}+\delta\boldsymbol{r}, t+\delta t) - Q(\boldsymbol{r}, t)}{\delta t}\right\}$$

$$\approx \lim_{\delta t \to 0} \left\{\frac{\delta t \frac{\partial Q}{\partial t} + \delta\boldsymbol{r}.\left\{\boldsymbol{\nabla} Q + \delta t \frac{\partial(\boldsymbol{\nabla} Q)}{\partial t}\right\}}{\delta t}\right\}$$

$$\approx \frac{\delta t}{\delta t}\frac{\partial Q}{\partial t} + \frac{\delta\boldsymbol{r}}{\delta t}.\left\{\boldsymbol{\nabla} Q + \delta t \frac{\partial(\boldsymbol{\nabla} Q)}{\partial t}\right\}$$

We can drop this off as this term is very small compared to the others.

$$\approx \frac{\partial Q}{\partial t} + \frac{\delta\boldsymbol{r}}{\delta t}.\boldsymbol{\nabla} Q$$

This is just a velocity!

The dot product shows the projection of $\boldsymbol{\nabla} Q$ onto the direction of motion of the fluid element (the projection of the spatial changes of Q).

Hence, if the flow velocity is **u**:

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla} Q$$

Lagrangian derivative (thermometer on the boat!)

Eulerian derivative (riverbed thermometer!)

"convective"/ "advective" derivative – due to the fact that the boat has moved to somewhere with a different temperature.

This equation is used to transform between Eulerian and Lagrangian derivatives.

* This step is to ensure that they're evaluated at the same time and is done again by Taylor expansion.

NOTE:

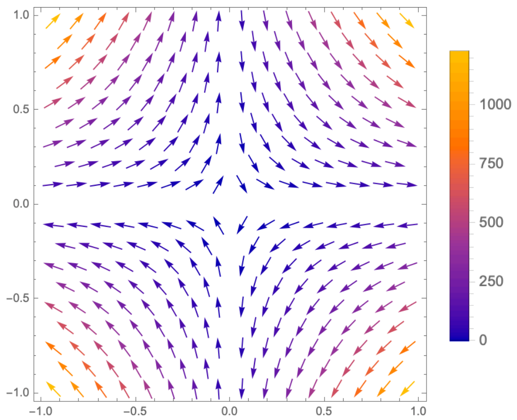
1. In steady state, $\frac{\partial Q}{\partial t} = 0$. The quantity Q, at the same place, does not change with time.
2. In steady state situations, Q will only be unchanging in time if the observer doesn't move. If the observer does move, then Q may or may not be unchanging.
3. $\frac{dQ}{dt} = 0$ ONLY if the situation is steady state and uniform.

Streamlines and stream functions:
If we want to visualise the flow of a fluid, we could do this in multiple ways. We might:

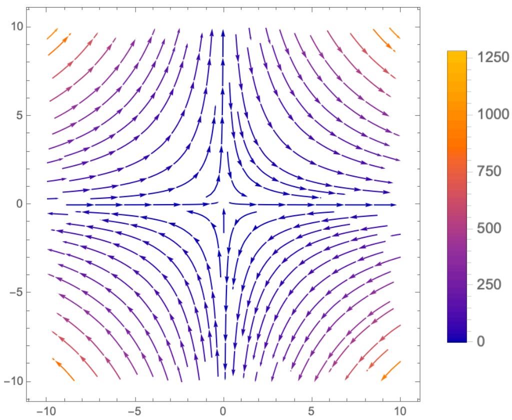
- Draw a vector plot, i.e. draw arrows which show the direction and magnitude of the velocity field, for example in the arrow size or colour.
- We could use **streamlines**, which are lines in the flow field that are tangential to the velocity everywhere.

Example: for the velocity field (or flow that can be described by) $\boldsymbol{u} = (x^2 y, -xy^2)$:

A vector plot looks like (a) in the figure below, where the colour represents the magnitude of the velocity. A plot of the streamlines is shown in (b).



(a)



(b)

You can think of streamlines as lines that trace out the flow of a fluid.

We can write a 2D flow $\boldsymbol{u}(x, y)$ in terms of a scalar ψ (known as a stream function) such that:

For an incompressible fluid, satisfying $\boldsymbol{\nabla}.\boldsymbol{u} = \boldsymbol{0}$, we can always represent $\boldsymbol{u}$ as:
$\boldsymbol{u} = -\boldsymbol{\nabla} \times (\psi \hat{\boldsymbol{z}})$
Since $\boldsymbol{\nabla}.\boldsymbol{u} = \boldsymbol{\nabla}.(\boldsymbol{\nabla} \times (\psi \hat{\boldsymbol{z}})) = \boldsymbol{0}, \quad \forall\, \psi$

We take ψ to be a function of x and y, then:
$\boldsymbol{u} = -\left(\frac{\partial \psi}{\partial y}\hat{\boldsymbol{x}} - \frac{\partial \psi}{\partial x}\hat{\boldsymbol{y}}\right)$

i.e., $u_x = -\frac{\partial\psi(x,y)}{\partial y}$ and $u_y = \frac{\partial\psi(x,y)}{\partial x}$

we may wish to do this because streamlines can be easier to plot than vector plots.

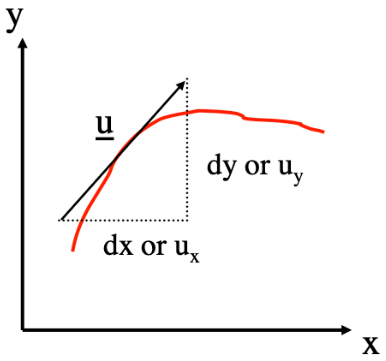
A streamline is defined as a curve that has u in the tangential direction. Along streamlines:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = 1$$

$$\longrightarrow u_x dy - u_y dx = 0$$

$$-\frac{\partial\psi}{\partial y}dy - \frac{\partial\psi}{\partial x}dx = 0$$

$d\psi = 0$ so ψ is a constant along streamlines.
We can therefore plot streamlines as a contour.



Streamfunctions. (ψ) are mathematical equations that describe the streamline in a flow.

Returning to our example of $\boldsymbol{u} = (x^2y, -xy^2)$, we can now go about plotting the streamlines (even though we've already seen the answer in (b) of the example, thanks to Mathematica's StreamPlot function!):

$$u_x = -\frac{\partial\psi(x,y)}{\partial y} \text{ and } u_y = \frac{\partial\psi(x,y)}{\partial x} \text{ so:}$$

$$\psi = -\int u_x\, dy = \int u_y\, dx$$

$$\psi = -\int u_x\, dy = \int yx^2\, dy = -\frac{x^2\, y^2}{2} + C(x)$$

$$\psi = \int u_y\, dx = \int -xy^2\, dx = -\frac{x^2\, y^2}{2} + D(y)$$

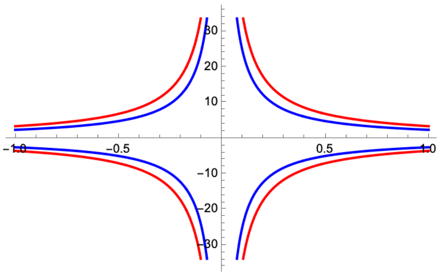
Where C and D are some constants. C and D must = 0.

$$\psi = -\frac{x^2\, y^2}{2}$$

Set this equal to some value eg

$$3 = -\frac{x^2\, y^2}{2}$$

Solve for x and plot…



The red curves show $\psi = 3$ and blue shows $\psi = 5$. The streamfunction ψ is a constant along a streamline but not across streamlines.

<u>Reminder:</u>

- If Q is a scalar:

$$\boldsymbol{u}.\boldsymbol{\nabla}Q = u_x\frac{\partial Q}{\partial x} + u_y\frac{\partial Q}{\partial y} + u_z\frac{\partial Q}{\partial z}$$

This will give you a scalar – both $\boldsymbol{u}$ and $\boldsymbol{\nabla}Q$ are vectors so the dot product will give a scalar.

- If, however, Q is a vector: $(\boldsymbol{u}.\boldsymbol{\nabla})$ is an operator which acts on $\boldsymbol{Q}$ i.e.

$$(\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{Q} = u_x\frac{\partial \boldsymbol{Q}}{\partial x} + u_y\frac{\partial \boldsymbol{Q}}{\partial y} + u_z\frac{\partial \boldsymbol{Q}}{\partial z}$$

$$(\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{Q} = \left(u_x\frac{\partial Q_x}{\partial x} + u_y\frac{\partial Q_x}{\partial y} + u_z\frac{\partial Q_x}{\partial z}\right)\hat{\boldsymbol{x}} + \left(u_x\frac{\partial Q_y}{\partial x} + u_y\frac{\partial Q_y}{\partial y} + u_z\frac{\partial Q_y}{\partial z}\right)\hat{\boldsymbol{y}} + \left(u_x\frac{\partial Q_z}{\partial x} + u_y\frac{\partial Q_z}{\partial y} + u_z\frac{\partial Q_z}{\partial z}\right)\hat{\boldsymbol{z}}$$

  This will give you a vector! The scalar operator $(\boldsymbol{u}.\boldsymbol{\nabla})$ acts on the vector $\boldsymbol{Q}$ and so, will give a vector!

<u>Question 1:</u> The temperature variation in a river is $T(x,y) = e^x \sin t$ and the river flows with velocity $\boldsymbol{u}(x,y) = (xt^2, 0)$. Write down both the Lagrangian and Eulerian derivatives.

<u>Answer 1:</u>
Eulerian:

$$\frac{\partial T}{\partial t} = e^x \cos t$$

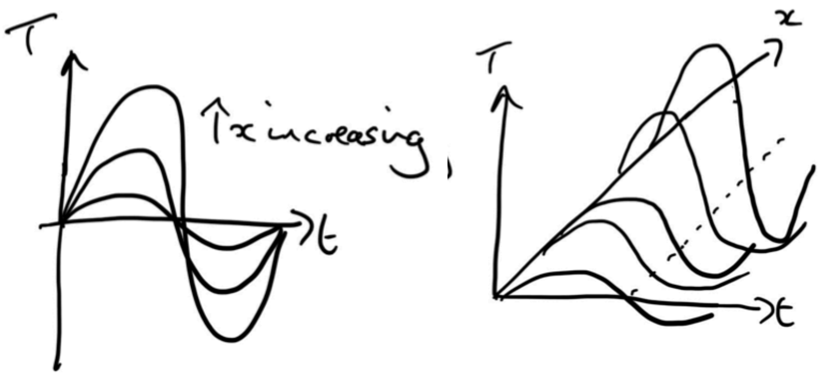
Lagrangian:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla}T) = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y}$$
$$= e^x \cos t + xt^2 e^x \sin t$$

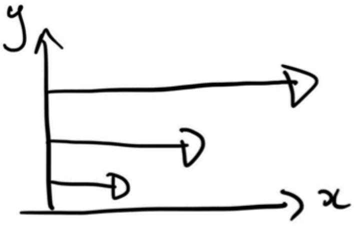
<u>Question 2:</u> An air flow has velocity $\boldsymbol{u}(x,y) = (y^2, 0)$ and temperature $T(x,t) = x\sin t$. Draw curves of $T(t)$ for a few values of x. Use this to help you draw a surface plot of $T(x,t)$. Then sketch the velocity vectors on the (x,y) plane and calculate the temperature variation with time:

(i) as seen at a fixed position,
(ii) as experience by a dust particle carried in the air.

<u>Answer 2:</u>
Drawing T(t) for a few values of x just involved sketching sine curves with various amplitudes. To convert this into a surface plot, we should draw another axis (x) and spread these sine curves along it.



The velocity vectors in the x-y plane can be sketched by noting that there is no component of velocity in the y direction, so all arrows should point parallel to the x-axis. The x-component of velocity increases with increasing y, so the arrows should get bigger as we increase y.



To calculate the variation in T with time:

(i) $\frac{\partial T}{\partial t} = x\cos t$

(ii) $\frac{dT}{dt} = \frac{\partial T}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla}T) = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y}$

$\frac{dT}{dt} = x\cos t + y^2\sin t$

<u>Question 3:</u> Sketch the 2D flow $\boldsymbol{u}(x, y) = (x, -y)$.

<u>Answer 3:</u>

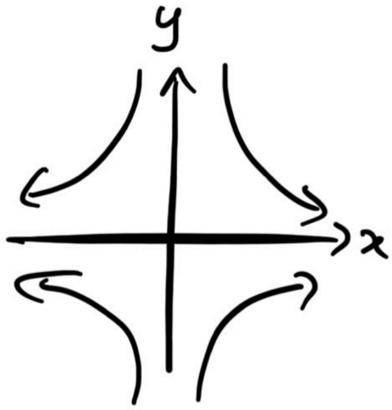
$$\frac{dx}{u_x} = \frac{dy}{u_y}$$

Therefore

$$\frac{dx}{x} = \frac{dy}{-y}$$

So

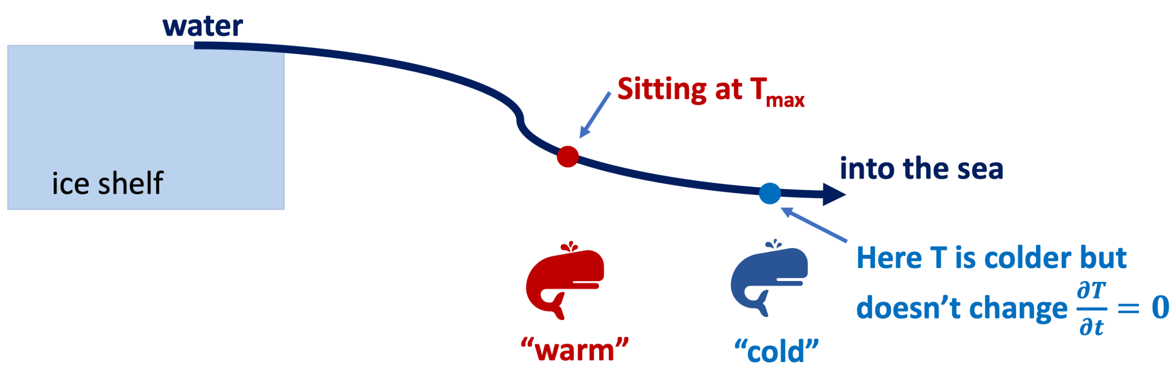
$$\int\frac{dx}{x} = \int\frac{dy}{-y}$$
$$\ln x = -\ln y + C$$
$$\ln x = -\ln y + \ln K$$
$$\ln x = \ln(y^{-1}) + \ln K$$
$$\ln x = \ln\left(\frac{K}{y}\right)$$
$$x = \frac{K}{y}$$



Where C and K are constants. This can then be sketched.

Lecture 3

<u>"Springtime in Antarctica":</u>



The ice melts in spring and water flows off the ice shelf into the sea. The water flows with a constant speed, u_x therefore $\boldsymbol{u} = (u_x, 0, 0)$.

This flow carries plankton with it. The whales eat the plankton, but only if alive and the plankton dies if the temperature starts to drop.

$$T = e^{kt} \sin x + T_0 \qquad \text{equation 1}$$

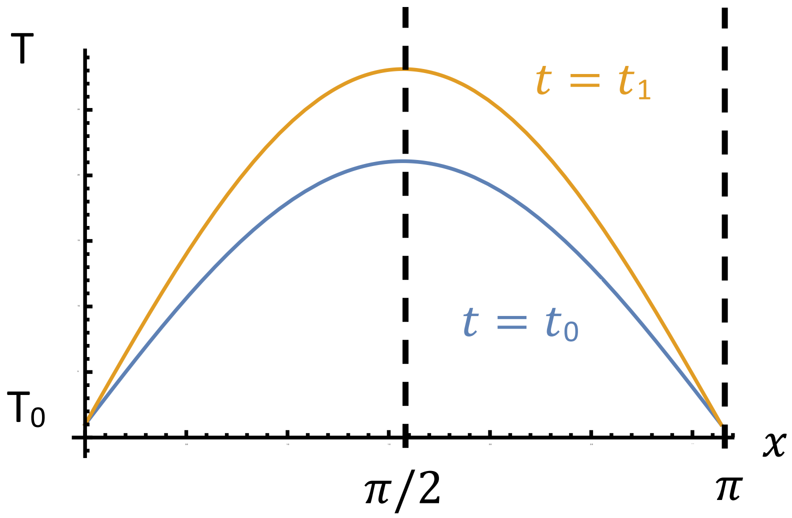
warming of water by the Sun, k > 0.

spatial variation of T

ambient T

Equation 1 describes the water temperature.

Drawing this function:



So, the temperature variation experienced by whales:

$$\frac{\partial T}{\partial t} = ke^{kt} \sin x$$

"cold" whales: $x = \pi$: $\frac{\partial T}{\partial t} = 0$ i.e. steady state.

"warm" whales: $x = \frac{\pi}{2}$: $\frac{\partial T}{\partial t} = ke^{kt}$

So, what is the temperature variation experienced by the plankton?

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + (\boldsymbol{u}.\boldsymbol{\nabla}T) = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + u_y\frac{\partial T}{\partial y} + u_z\frac{\partial T}{\partial z}$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u_x\frac{\partial T}{\partial x} + 0 + 0$$

$$\frac{dT}{dt} = ke^{kt} \sin x + u_x\frac{\partial T}{\partial x}$$

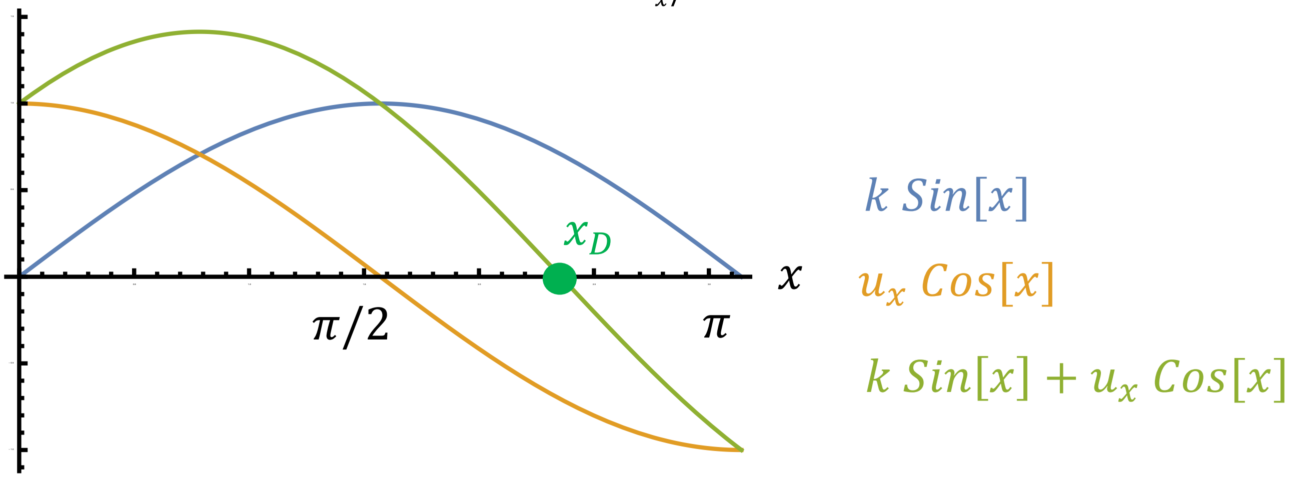
$$\frac{dT}{dt} = ke^{kt} \sin x + u_x e^{kt} \cos x$$

$$\frac{dT}{dt} = e^{kt}(k \sin x + u_x \cos x)$$

When $\frac{dT}{dt} < 0$, the temperature is dropping for the plankton, and they die. Then the whales are hungry!

When does $\frac{dT}{dt} = 0$?

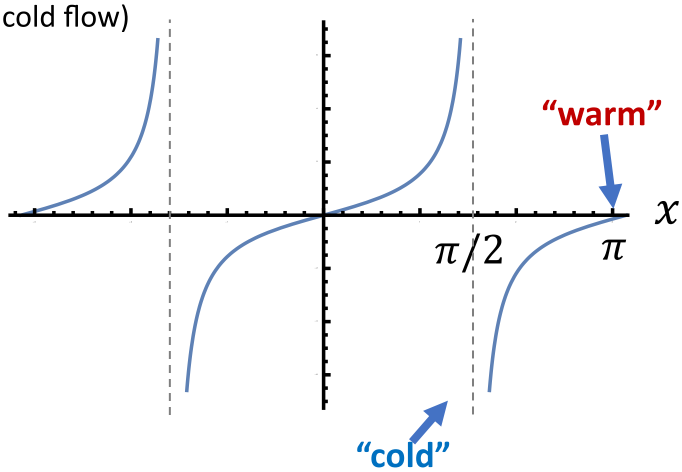
$$k \sin x = -u_x \cos x$$
$$\tan x = -u_x/k$$



NOTE: u_x and k are positive constants, therefore, we are looking for negative values of $\tan x$.

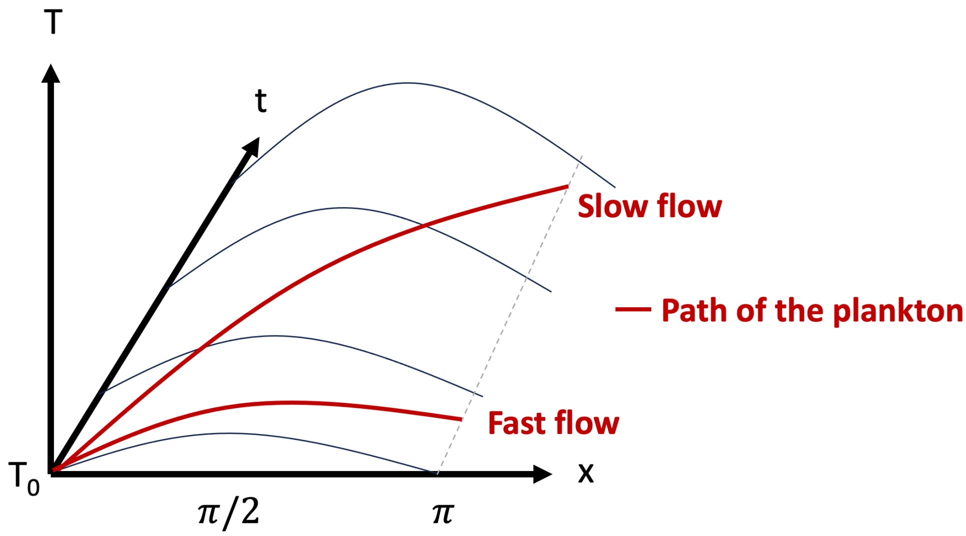
Take $u_x/k \longrightarrow 0$: slow flow, u_x is small so $x_D = \pi$ (the warm flow)
Take $u_x/k \longrightarrow \infty$: fast flow, u_x is big so $x_D = \frac{\pi}{2}$ (the cold flow)

We've modelled a rate of flow: u_x
We've modelled a rate of heating: k



The ratio of u_x/k is a ratio of two rates. Depending on if it's a large or small ratio, the plankton will travel a short or long distance before they die.

Pictorially:



NOTE: $u_x = x/t$

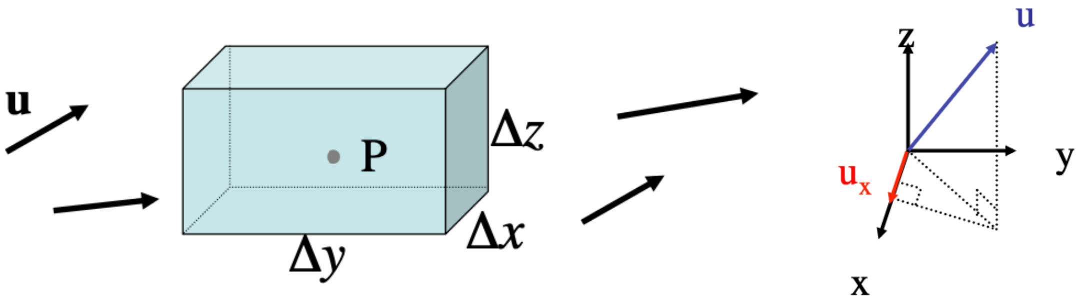
- For a slow flow, it takes a long time to travel the distance $x = 0$ to $x = \pi$.
- For a fast flow, it takes a short time to travel this distance.

Plankton can travel different paths over the surface. Thinking about the temperature variation over the surface – i.e. experienced by the plankton in the different paths:

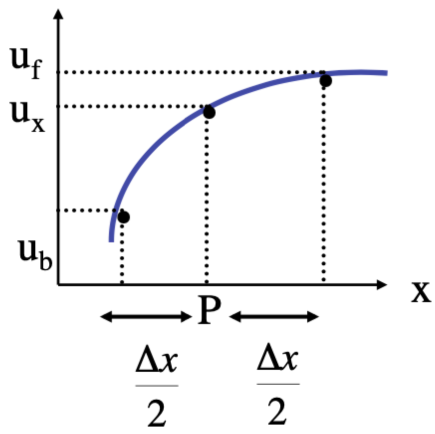
- Fast flow: plankton reach $x = \pi$ before the water temperature has had time to increase.
- Slow flow: plankton experience a rising temperature because the water is heated, not because of the flow.

The graph shows the difference between the temperature variation experienced by the whales vs the plankton. i.e. Eulerian vs Lagrangian.

<u>Aside on divergence:</u>
Consider a gas flow $\boldsymbol{u}$ through a box with centre P.



At P, the gas has velocity $\boldsymbol{u} = (u_x, u_y, u_z) = \boldsymbol{u}(\boldsymbol{r}, t)$. Now consider how the velocity varies away from this point.



$$f(x) = f(a) + (x-a)f'(a) + \cdots$$

At the back face: $u_b \approx u_x - \frac{\Delta x}{2}\frac{\partial u_x}{\partial x}$

At the front face: $u_f \approx u_x + \frac{\Delta x}{2}\frac{\partial u_x}{\partial x}$

Using a Taylor expansion and keeping only linear terms.

Evaluate u_x at P

Volume of gas crossing back face per second:

$$\frac{V}{t} = \frac{d.A}{t}$$

Volume/second = distance moved/second * area of the face

$$= (u_x - \frac{1}{2}\frac{\partial u_x}{\partial x}\Delta x)\Delta y \Delta z$$

Volume of gas crossing front face per second:

$$= (u_x + \frac{1}{2}\frac{\partial u_x}{\partial x}\Delta x)\Delta y \Delta z$$

Net volume/second in the x direction $= \frac{\partial u_x}{\partial x}\Delta x \Delta y \Delta z$

By analogy, for the y direction $= \frac{\partial u_y}{\partial y}\Delta x \Delta y \Delta z$

for the z direction $= \frac{\partial u_z}{\partial z}\Delta x \Delta y \Delta z$

Total net volume/second:

$$V_T/_{sec} = \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

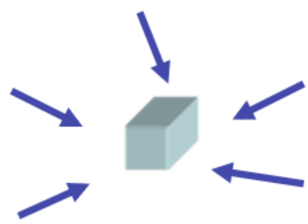
$$V_T/_{sec} = (\boldsymbol{\nabla}.\boldsymbol{u})\ \Delta x \Delta y \Delta z$$

Divergence of $\boldsymbol{u}$ gives the volume of the gas emerging per second from a unit volume.

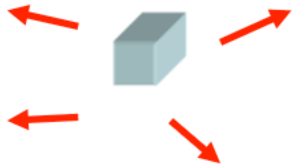
Multiply by the volume element.

The divergence shows how the flux of the gas behaves:

$(\boldsymbol{\nabla}.\boldsymbol{u}) < 0$: Gas flows in (**SINK**)



$(\boldsymbol{\nabla}.\boldsymbol{u}) > 0$: Gas flows out (**SOURCE**)

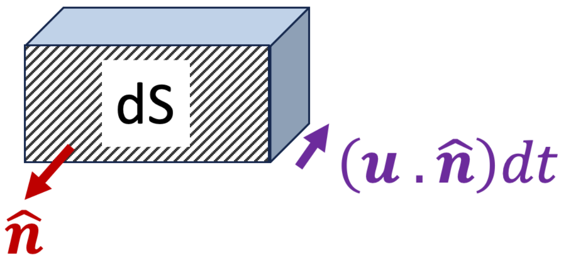


We can find the total flux:

$$\Phi = \iiint_V (\boldsymbol{\nabla}.\boldsymbol{u})\, dV$$

(integrate $(\boldsymbol{\nabla}.\boldsymbol{u})$ over volume)

We could, however, think of it as the amount of fluid that crosses each surface element, dS, in time dt.



Distance swept out by the surface $= \boldsymbol{u}\,.\,\hat{\boldsymbol{n}}\, dt$
Volume that flows through dS each second: $= (\boldsymbol{u}\,.\,\hat{\boldsymbol{n}})\, dS$
Add up all of these: $\iint_S \ \boldsymbol{u}.\, d\boldsymbol{S}$

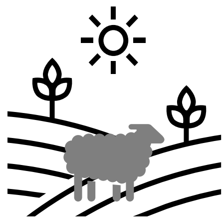
These must be equal so:

$$\boxed{\Phi = \oiint_S \ \boldsymbol{u}.\, d\boldsymbol{S} = \iiint_V (\boldsymbol{\nabla}.\boldsymbol{u})\, dV}$$

DIVERGENCE THEOREM!

$d\boldsymbol{S} = dS\, \hat{\boldsymbol{n}}$

This applies to *any* flux, water in a hose pipe, magnetic field on stars, sheep in fields…

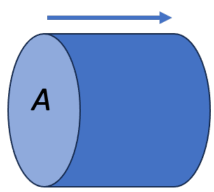


How do we interpret this for a fluid? It tells us the volume of fluid emerging per second.

This emerging volume carries mass. The fluid flows at speed u through an area A. The volume swept out per second by this flow is uA:
$\text{volume / second} \ = \ u\, A$
$\text{density} = \rho = \text{mass / volume}$

So $\text{mass / second} \ = \ (\text{mass / volume})\ (\text{volume / second}) \ = \ \rho\, u\, A$

Distance travelled / second = u



If $A = 1$ (i.e. unit area)

$$\frac{mass}{second}\, through\ unit\ area = mass\ flux = \ \rho u$$

Question 4a: If a steady-state, 2D velocity field $\boldsymbol{u}$ is divergence-free and $u_x = a(x^2 - y^2)$ for some constant a, what must u_y be?

Answer 4a:

$$\boldsymbol{\nabla}.\boldsymbol{u} = 0$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$

$$\frac{\partial u_y}{\partial y} = -\frac{\partial u_x}{\partial x} = -a(2x)$$

$$u_y = \int -2ax\, dy$$

$$u_y = -2axy + C(x)$$

Where C is an integration constant that may depend on x.

Question 4b: Determine a form for the stream function for this flow if C(x)=0.

Answer 4b:

$$u_x = -\frac{\partial \psi(x,y)}{\partial y} \text{ and } u_y = \frac{\partial \psi(x,y)}{\partial x} \text{ so:}$$
$$\psi = -\int u_x\, dy = \int u_y\, dx$$
$$\psi = -\int u_x\, dy = \int a(x^2 - y^2)\, dy = -a(x^2 y - \frac{y^3}{2}) + D(x)$$
$$\psi = \int u_y\, dx = \int -2axy\, dx = -2ay\frac{x^2}{2} + E(y) = -ayx^2 + E(y)$$

Comparing these:

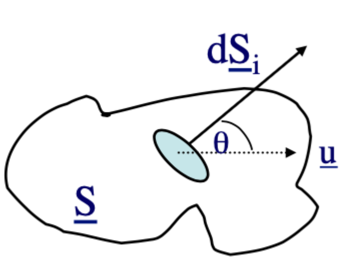
$$\psi = -ax^2 y + a\frac{y^3}{2} + D(x) = -ayx^2 + E(y)$$

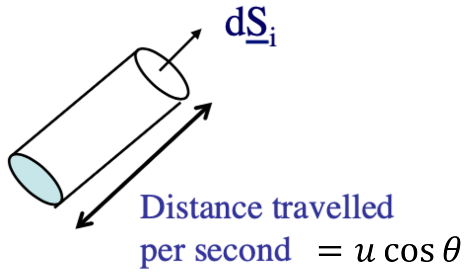
$D(x) = 0$ and $E(y) = a\frac{y^3}{2}$.

$$\psi = -ayx^2 + a\frac{y^3}{2} = ay\left(x^2 - \frac{y^2}{2}\right)$$

Lecture 4

Consider a volume V whose surface **S** is a patchwork of surface elements d**S**. A flow u through the surface has a component along the outward normal $= u\cos\theta$ (i.e. the projection onto that particular outward normal of the velocity).





Every second:

- This flow travels a distance $u\cos\theta$ in the direction of d**S** i.e. length = $u\cos\theta$.
- A mass, $m = \rho V = \rho \times length \times area$ passes through the surface d**S**.

$$m = \rho V = \rho \times length \times area = \rho\, u \cos\theta\, dS = \rho\, \boldsymbol{u} \,.\, d\boldsymbol{S}$$

Recall: $u\cos\theta = \boldsymbol{u}.\frac{d\boldsymbol{S}}{|d\boldsymbol{S}|}$

The total rate at which mass flows through the surface **S** is the sum of all the elements:

$$-\sum_i \rho\, \boldsymbol{u} \,.\, d\boldsymbol{S}_i = -\oiint_S \rho\, \boldsymbol{u} \,.\, d\boldsymbol{S} = -\iiint_V \boldsymbol{\nabla} \,.\, (\rho \boldsymbol{u})\, dV$$

Negative because
it is outward flow!

In the absence of sources or sinks of mass, this must be equal to the rate of change of mass of fluid in V:

$$\frac{\partial M}{\partial t} in\ V = \iiint_V \frac{\partial \rho}{\partial t}\, dV = \iiint_V \frac{\partial}{\partial t}(\rho dV)$$

This is a mass, because
$m = \rho V$

Recall: can always swap
spatial and temporal
terms because they don't
depend on each other.

This is a partial because it's
rate of change of mass
from a fixed point.

i.e. $\frac{\partial}{\partial t}\iiint_V \rho dV \overset{!}{=} -\iiint_V \boldsymbol{\nabla} \,.\, (\rho \boldsymbol{u})\, dV$

NOTE: $\overset{!}{=}$ is used here
to mean "must equal".

This is for the same volume so:

$$\frac{\partial}{\partial t}\iiint_V \rho dV + \iiint_V \boldsymbol{\nabla} \,.\, (\rho \boldsymbol{u})\, dV = 0$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \,.\, (\rho \boldsymbol{u})\right) dV = 0, \quad \forall\ dV$$

Hence:

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \,.\, (\rho \boldsymbol{u}) = 0}$$

Eulerian form of the continuity
equation/conservation of mass.

We could also write the Lagrangian form:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}\rho$$
$$\frac{d\rho}{dt} = -\boldsymbol{\nabla}.(\rho\boldsymbol{u}) + \boldsymbol{u}.\boldsymbol{\nabla}\rho$$
$$\frac{d\rho}{dt} = -\rho(\boldsymbol{\nabla}.\boldsymbol{u}) - \boldsymbol{u}.\boldsymbol{\nabla}\rho + \boldsymbol{u}.\boldsymbol{\nabla}\rho$$
$$\boxed{\frac{d\rho}{dt} + \rho(\boldsymbol{\nabla}.\boldsymbol{u}) = 0}$$

Lagrangian form of the continuity equation/conservation of mass.

NOTE:

(1) The definition of incompressible flows: $\frac{d\rho}{dt} = 0$. So, density is in steady state *and* uniform.
(2) This implies that $\boldsymbol{\nabla}.\boldsymbol{u} = 0$ for incompressible fluids, i.e. divergence free.
(3) $\frac{d\rho}{dt} = 0$ for fluid elements. Density need not be conserved overall.
(4) $\boldsymbol{\nabla}.\boldsymbol{u} = 0$ can be very useful: only need one component to get the other in 2D!

Question 5: If you have a steady, incompressible 2D flow $\boldsymbol{u} = (u_x, u_y)$ where $u_y = -\sinh y$, use the equation of conservation of mass to get u_x.

Answer 5:

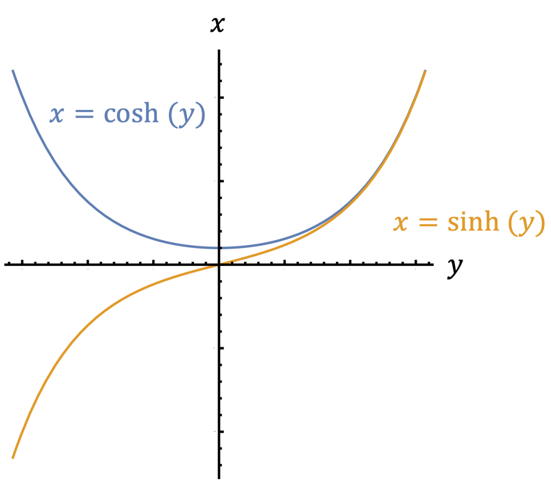
Steady state so: $\frac{\partial\rho}{\partial t} = 0$

Incompressible so: $\frac{d\rho}{dt} = 0$ => $\boldsymbol{\nabla}.\boldsymbol{u} = 0$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$$
$$-\frac{\partial u_x}{\partial x} = \frac{\partial u_y}{\partial y} = \frac{\partial(-\sinh y)}{\partial y} = -\cosh y$$
$$\frac{\partial u_x}{\partial x} = \cosh y$$
$$u_x = \int \cosh y \, dx = x\cosh y + u_x(0)$$

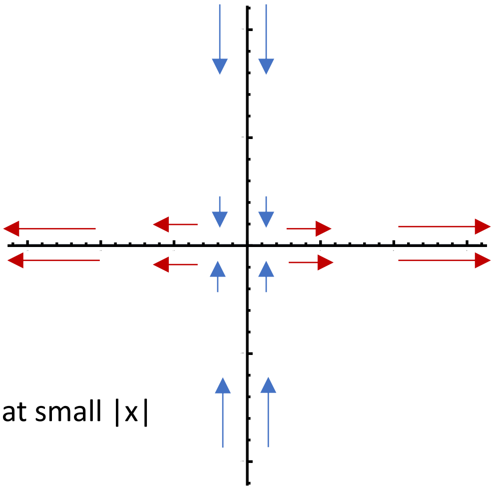
Drawing: if $u_x(0) = 0$...
$u_x = x\cosh y$ so $\boldsymbol{u} = (x\cosh y, -\sinh y)$.

Reminder:

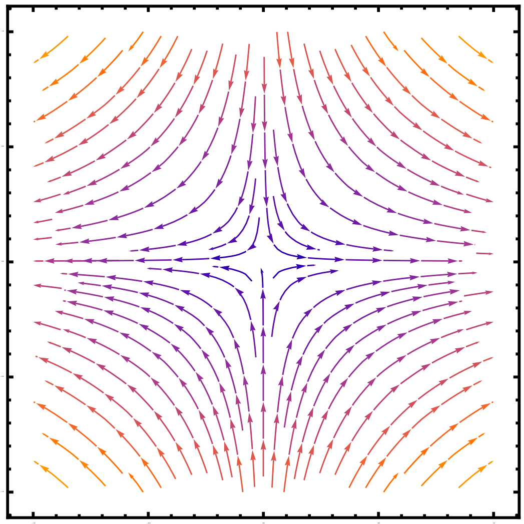


Plotting the vectors:

- Along the y axis where x = 0, $\boldsymbol{u} = (0, -\sinh y)$.
    - So, when y is positive it points towards the origin,
    - when y is negative it points towards the origin.
    - Its magnitude is bigger at large y than small y.
- Along the x axis where y = 0
    - $\boldsymbol{u} = (0,0)$.
- Along any other constant y, u_x is big at large |x| and small at small |x|



If we put this all together, we get:



More elegantly:

$$\frac{dx}{u_x} = \frac{dy}{u_y}$$ (streamline)

i.e.

$$\frac{u_y}{u_x} = \frac{dy}{dx} = -\frac{\sinh y}{x \cosh y}$$

$$\int \frac{\cosh y}{\sinh y} dy = -\int \frac{1}{x} dx$$

$$\ln(\sinh y) = -\ln x + c$$

$$\ln(\sinh y) = \ln\left(\frac{1}{x}\right) + c$$

$$x \sim \frac{1}{\sinh y}$$ (ignoring the constant)

<u>Question 6</u>: Under what condition does this velocity field represent an incompressible flow that conserves mass?

$$\boldsymbol{u} = (u_x, u_y, u_z)$$

Where $u_x = a_1x + b_1y + c_1z$, $u_y = a_2x + b_2y + c_2z$ and $u_z = a_3x + b_3y + c_3z$ and as bs and cs are constants.

Answer 6:

Conservation of mass: $\frac{d\rho}{dt} + \rho(\boldsymbol{\nabla}.\boldsymbol{u}) = 0$

And incompressible so $\boldsymbol{\nabla}.\boldsymbol{u} = 0$ i.e.:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial}{\partial x}(a_1 x + b_1 y + c_1 z) + \frac{\partial}{\partial y}(a_2 x + b_2 y + c_2 z) + \frac{\partial}{\partial z}(a_3 x + b_3 y + c_3 z) = 0$$

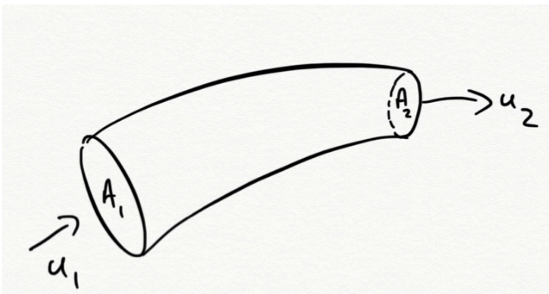
$b_1, c_1, a_2, c_2, a_3, b_3$ can take any value, since they are constants and these terms will = 0 when the partials are taken. Leaving:

$$\frac{\partial}{\partial x}(a_1 x) + \frac{\partial}{\partial y}(b_2 y) + \frac{\partial}{\partial z}(c_3 z) = 0$$

$$a_1 + b_2 + c_3 = 0$$

At least one of these constants must be negative, so that these can sum to 0.

Question 7: Consider steady incompressible flow along a tube. The speed is u_1 at A_1 and u_2 at A_2. If $A_1 = 1m^2$ what must A_2 be in order that the flow doubles in speed by the time it reaches A_2?



Answer 7:

What can we say about this scenario?

Steady state: $\frac{\partial}{\partial t} = 0$ so the conservation of mass equation simplifies:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}.(\rho \boldsymbol{u}) = 0$$

$$\boldsymbol{\nabla}.(\rho \boldsymbol{u}) = 0$$

And also, incompressible so $\boldsymbol{\nabla}.\boldsymbol{u} = 0$:

$$\boldsymbol{\nabla}.(\rho \boldsymbol{u}) = \rho \boldsymbol{\nabla}.\boldsymbol{u} + \boldsymbol{u}.\boldsymbol{\nabla}\rho = \boldsymbol{u}.\boldsymbol{\nabla}\rho = 0$$

Using the divergence theorem: $\oiint_S \boldsymbol{u}.d\boldsymbol{S} = \iiint_V \boldsymbol{\nabla}.\boldsymbol{u}\, dV$

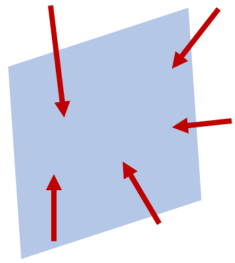
$$u_2 A_2 - u_1 A_1 = 0$$

Therefore, if $u_2 = 2u_1$ then $A_2 = \frac{u_1 A_1}{u_2} = \frac{A_1}{2}$

Conservation of momentum:
Rate of change of momentum = sum of forces

What are the forces acting on a parcel of fluid?

- Put a surface in a fluid and there is a momentum flux across it (one from each side).



This is purely a consequence of thermal properties, NOT related to bulk properties!
These are small scale thermal velocities – i.e. they cancel out from both sides, but we are considering only one.

Microscopically (in a perfect gas):

- Finite temperature imparts molecules with random motions.
- The pressure is the associated (one sided) momentum flux.

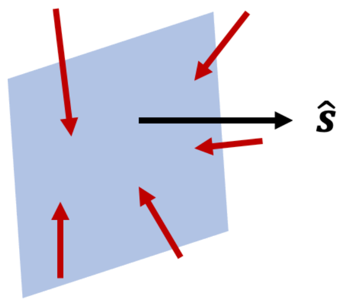
(If we have a truly isothermal gas…)
Since these motions are isotropic, the momentum flux locally is:

- Independent of surface orientation.
- Always perpendicular to the surface (parallel ones aren't carrying momentum through it).

Quick check on units:

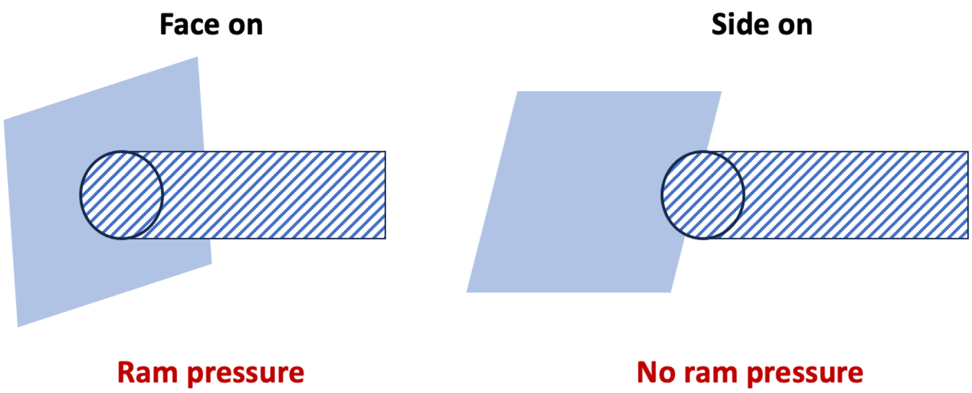
- Pressure is a force per area: $p = F/A$
- Momentum flux is the rate of flow of momentum: $\frac{momentum/second}{A}$
- Force is a rate of change of momentum.

Pressure:
Consider forces between elements: for collisional fluids, this means there are forces between particles/within the field – these forces relate to local temperature. Taking any surface placed within a fluid, there is a momentum flux across it – particle travel through the surface due to microscopic properties.



NOTE:

(1) The thermal pressure is associated with random motions in the fluid which are isotropic. It is a scale.
(2) The ram pressure is associated with bulk motions of the fluid – only a surface whose normal has some component along the direction of the flow feels the ram pressure.



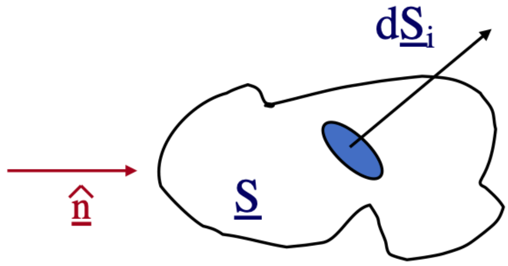
Lecture 5

Consider a lump of fluid subject to gravity and the inward pressure of the surrounding fluid. We care about forces acting along the direction of $\hat{\boldsymbol{n}}$:

Pressure is a force/area:

$$\boldsymbol{F}_{int} = -pd\boldsymbol{S}_i$$

This is negative because we care about pressure in and $d\boldsymbol{S}_i$ points out.



The component along $\hat{\boldsymbol{n}}$ is:

$$\boldsymbol{F}_{int} \,.\, \hat{\boldsymbol{n}} = -pd\boldsymbol{S}_i \,.\, \hat{\boldsymbol{n}}$$

This is the force the fluid element feels due to its environment.

and we care about all of these, therefore integrating:

$$\boldsymbol{F}_{int} \,.\, \hat{\boldsymbol{n}}_{total} = -\oiint_S pd\boldsymbol{S}_i \,.\, \hat{\boldsymbol{n}} = -\oiint_S p\,\hat{\boldsymbol{n}} \,.\, d\boldsymbol{S}_i = -\iiint_V \boldsymbol{\nabla} \,.\, (p\hat{\boldsymbol{n}})dV$$

we also have an externally applied force:

$$\boldsymbol{F}_{ext} \,.\, \hat{\boldsymbol{n}} = \iiint_V \rho\boldsymbol{g} \,.\, \hat{\boldsymbol{n}}\, dV \qquad [\, mg = \rho g V \,]$$

here this is just gravity, though we could also add other forces eg viscosity, magnetic forces etc.

The total momentum in the volume V is:

$$\iiint_V \rho\boldsymbol{u}\, dV$$

The rate of change of this along $\hat{\boldsymbol{n}}$ is:

$$\left(\frac{d}{dt}\iiint_V \rho\boldsymbol{u}\, dV\right) \,.\, \hat{\boldsymbol{n}}$$

Hence, the equation of motion (EOM) in direction $\hat{\boldsymbol{n}}$ is:

$$\underbrace{\left(\frac{d}{dt}\iiint_V \rho\boldsymbol{u}\, dV\right) \,.\, \hat{\boldsymbol{n}}}_{\text{Lagrangian derivative. Momentum contained in volume V.}} = \underbrace{-\iiint_V \boldsymbol{\nabla} \,.\, (p\hat{\boldsymbol{n}})dV}_{\text{“pressure force”}} + \underbrace{\iiint_V \rho\boldsymbol{g} \,.\, \hat{\boldsymbol{n}}\, dV}_{\text{“mg”}}$$

But NOTE: $\nabla.(p\hat{\boldsymbol{n}}) = p\nabla.\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}}.\nabla p = \hat{\boldsymbol{n}}.\nabla p$ $\qquad$ $\nabla.\hat{\boldsymbol{n}} = 0$, as $\hat{\boldsymbol{n}}$ is a unit vector.

Assuming the lump is small, $\int dV \rightarrow \delta V$:

$$\left(\frac{d}{dt}\,\rho\boldsymbol{u}\,\delta V\right).\,\hat{\boldsymbol{n}} = -\boldsymbol{\nabla}.\,(p\hat{\boldsymbol{n}})\delta V + \rho\boldsymbol{g}\,.\hat{\boldsymbol{n}}\,\delta V$$

$$\left(\frac{d}{dt}\,\rho\boldsymbol{u}\,\delta V\right).\,\hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}.\nabla p\,\delta V + \rho\boldsymbol{g}\,.\hat{\boldsymbol{n}}\,\delta V$$

$$\left(\frac{d\boldsymbol{u}}{dt}\,(\rho\,\delta V) + \frac{d(\rho\,\delta V)}{dt}\,\boldsymbol{u}\right).\,\hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}.\nabla p\,\delta V + \rho\boldsymbol{g}\,.\hat{\boldsymbol{n}}\,\delta V$$

= 0 because the mass of the lump is conserved.

$$\left((\rho\,\delta V)\frac{d\boldsymbol{u}}{dt}\,.\,\hat{\boldsymbol{n}} + \cancel{\frac{d(\rho\,\delta V)}{dt}}\,\boldsymbol{u}.\,\hat{\boldsymbol{n}}\right) = -\hat{\boldsymbol{n}}.\nabla p\,\delta V + \rho\boldsymbol{g}\,.\hat{\boldsymbol{n}}\,\delta V$$

$$(\rho\,\delta V)\frac{d\boldsymbol{u}}{dt}\,.\,\hat{\boldsymbol{n}} = -\hat{\boldsymbol{n}}.\nabla p\,\delta V + \rho\boldsymbol{g}\,.\hat{\boldsymbol{n}}\,\delta V$$

$\forall, \hat{\boldsymbol{n}}$ and δV

Hence,

$$\rho\frac{d\boldsymbol{u}}{dt} = -\nabla p + \rho\boldsymbol{g}$$

We can generalise this to include any external force:

$$\boxed{\rho\frac{d\boldsymbol{u}}{dt} = -\nabla p + \rho\boldsymbol{g} + \boldsymbol{F}_{ext}}$$

Lagrangian form of the EOM aka conservation of momentum.

What does this tell us? The momentum of a fluid element changes in response to pressure gradients and gravitational forces (and other external forces).

∇p drives the fluid i.e. gradients in pressure.

We can also find the Eulerian form of the EOM / conservation of momentum equation:

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \boldsymbol{u}.\boldsymbol{\nabla}Q$$

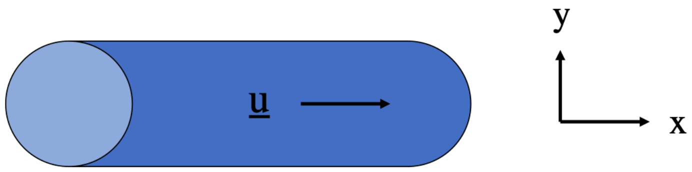
$$-\nabla p + \rho\boldsymbol{g} = \rho\frac{\partial\boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u}$$

$$\boxed{\rho\frac{\partial\boldsymbol{u}}{\partial t} = -\nabla p + \rho\boldsymbol{g} - \rho(\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u}}$$

Eulerian form of the EOM aka conservation of momentum.

The momentum contained in a fixed grid cell changes as a result of pressure gradients, gravitational forces <u>plus</u> any imbalance in momentum in and out of a cell.

Example: consider a flow $\boldsymbol{u} = u_x\hat{x}$ along a pipe in the absence of gravity.



$$-\nabla p = \rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{u}$$

the component along the hose is:

$$-\frac{\partial p}{\partial x} = \rho\frac{\partial u_x}{\partial t} + \rho\left(u_x\frac{\partial}{\partial x} + \cancelto{=0}{u_y\frac{\partial}{\partial y}} + \cancelto{=0}{u_z\frac{\partial}{\partial z}}\right)u_x$$

$$-\frac{\partial p}{\partial x} = \rho\frac{\partial u_x}{\partial t} + \rho u_x\frac{\partial u_x}{\partial x}$$

For an incompressible fluid, $\frac{\partial \rho}{\partial t} = 0$ so this equation becomes:

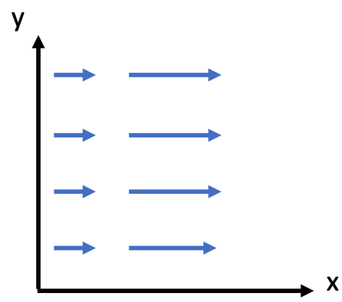
$$\underbrace{\frac{\partial(\rho u_x)}{\partial t}}_{\text{Rate of change of momentum}} + \underbrace{\frac{\partial}{\partial x}\left(\frac{\rho {u_x}^2}{2}\right)}_{\text{Ram pressure}} = \underbrace{-\frac{\partial p}{\partial x}}_{\text{Thermal pressure}}$$

Ram pressure is due to bulk motion of the fluid.

Question 5: Consider $\boldsymbol{u}(x, y) = (u_x, u_y) = (x, 0)$. Determine for a steady state, the variation in density and pressure, in the absence of gravity.

Answer 5:

Steady state therefore $\frac{\partial Q}{\partial t} = 0$.



Conservation of mass: $\boldsymbol{\nabla}.(\rho\boldsymbol{u}) = 0$.

$$\boldsymbol{\nabla}.(\rho\boldsymbol{u}) = \rho\boldsymbol{\nabla}.\boldsymbol{u} + \boldsymbol{\nabla}\rho.\boldsymbol{u} = 0$$

$$\rho\frac{\partial u_x}{\partial x} + u_x\frac{\partial \rho}{\partial x} = 0$$

$$\rho\frac{\partial x}{\partial x} + x\frac{\partial \rho}{\partial x} = 0$$

$$\rho + x\frac{\partial \rho}{\partial x} = 0$$

$$x\frac{\partial \rho}{\partial x} = -\rho$$

$$\int\frac{\partial \rho}{\rho} = -\int x\partial x$$

$$\ln\rho = -\ln x + k(y)$$

k(y) is a constant, which may depend on y.

$$\ln\rho = \ln\frac{c(y)}{x}$$

c(y) is a new constant, formed by bringing k(y) into the logarithm i.e. rewriting the constants as:

$$k(y) = \ln(c(y))$$

$$\rho = \frac{c(y)}{x}$$

Conservation of momentum: $\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\,.\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p$

$$\rho(0) + \rho(\boldsymbol{u}\,.\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p$$

$$\rho\left(u_x\frac{\partial}{\partial x} + \cancelto{=0}{u_y\frac{\partial}{\partial y}}\right)\boldsymbol{u} = -\boldsymbol{\nabla}p$$

$$\rho\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)(x\hat{\boldsymbol{x}} + 0\hat{\boldsymbol{y}}) = -\frac{\partial p}{\partial x}\hat{\boldsymbol{x}} - \frac{\partial p}{\partial y}\hat{\boldsymbol{y}}$$

x-component:

$$\rho\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)(x\hat{\boldsymbol{x}}) = -\frac{\partial p}{\partial x}\hat{\boldsymbol{x}}$$

$$\rho\left(x\frac{\partial x}{\partial x}\right) = -\frac{\partial p}{\partial x}$$

$$\rho x = -\frac{\partial p}{\partial x}$$

And we know that $\rho = \frac{c(y)}{x}$ from the conservation of mass so:

$$\frac{\partial p}{\partial x} = -c(y)$$

$$\int dp = -\int c(y)dx$$

$$p = -c(y)x + A(y)$$ A(y) is constant which may depend on y.

y-component:

$$\rho\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y}\right)0\hat{\boldsymbol{y}} = -\frac{\partial p}{\partial y}\hat{\boldsymbol{y}}$$

$$0 = -\frac{\partial p}{\partial y}$$

$$\int dp = \int 0\,dy$$

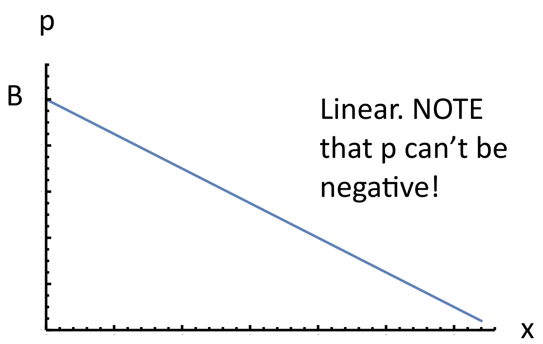
$$p = B(x)$$

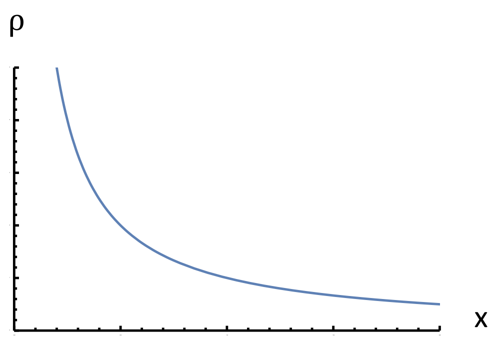
So, combining,

$$p = -Ax + B$$

Where A and B are some positive constants.

<u>Graphically:</u>





Lecture 6

The energy equation:
From the second law of thermodynamics:

$$T\,dS = \underbrace{dQ}_{\text{Heat exchange}} = \underbrace{dU}_{\text{Internal energy change}} + \underbrace{p\,dV}_{\text{Work done by the fluid}}$$

We can write this for a small mass element:

$$T\,\delta m\,ds = \delta m\,dq = \delta m\,de + p\,\underbrace{\delta m\,d\left({}^{1}\!/_{\rho}\right)}_{\text{Volume of unit mass} \times \delta m}$$

Where $e = \frac{p}{(\gamma-1)\rho}$ (from $p = \frac{R\rho T}{\mu}$; $p = (c_p - c_v)\rho T$) e is the internal energy per unit mass
And $\gamma = c_p/c_v$ i.e. the ratio of specific heats.

The small mass elements can then be cancelled:

$$T\,ds = dq = de + p\,d\left({}^{1}\!/_{\rho}\right)$$

And dividing by dt:

$$T\frac{ds}{dt} = \frac{dq}{dt} = \frac{de}{dt} + p\frac{d\left({}^{1}\!/_{\rho}\right)}{dt}$$

$$\rho\,T\frac{ds}{dt} = \rho\left[\frac{de}{dt} + p\frac{d\left({}^{1}\!/_{\rho}\right)}{dt}\right] = \rho\frac{dq}{dt} \equiv \underbrace{-L}$$

The sum of sources and sinks of energy. This is the energy loss function. <u>The negative is a convention.</u>

The system is not necessarily closed, so it could gain or lose energy. But we'll often have closed ones, where L = 0

$$\rho\frac{de}{dt} + \rho\,p\frac{d\left({}^{1}\!/_{\rho}\right)}{dt} = -L$$

$$\rho\frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = -L$$

And using mass conservation: $\frac{d\rho}{dt} = -\rho(\boldsymbol{\nabla}.\boldsymbol{u})$

$$\rho\frac{de}{dt} - \frac{p}{\rho}\left(-\rho(\boldsymbol{\nabla}.\boldsymbol{u})\right) = -L$$

$$\rho\frac{de}{dt} + p\boldsymbol{\nabla}.\boldsymbol{u} = -L \quad (*)$$

<u>NOTE</u>: if we have a closed system and $\boldsymbol{\nabla}.\boldsymbol{u} = 0$, either $\rho = 0$ or $\frac{de}{dt} = 0$ (no change in internal energy).

BUT! We wanted to include all energies and so far, we've just included the internal/thermal energies. We haven't accounted for any mechanical/bulk energies.
Examples of useful bulk energies are gravitation, kinetic energy, magnetic energies etc.

So now take:

to get the mechanical energy:

$$\boldsymbol{u}\,.\underbrace{\left(\rho\frac{d\boldsymbol{u}}{dt}+\boldsymbol{\nabla}p-\rho\boldsymbol{g}=\boldsymbol{0}\right)}_{\text{EOM/ conservation of momentum}}$$

$$\rho\,\boldsymbol{u}\,.\frac{d\boldsymbol{u}}{dt}+\boldsymbol{u}\,.\boldsymbol{\nabla}p-\rho\boldsymbol{u}\,.\boldsymbol{g}=0$$

And add this to (*). In adding them we have now combined both energy types (thermal energy and bulk energy)

$$\rho\frac{de}{dt}+p\boldsymbol{\nabla}.\boldsymbol{u}+\rho\,\boldsymbol{u}\,.\frac{d\boldsymbol{u}}{dt}+\boldsymbol{u}\,.\boldsymbol{\nabla}p-\rho\boldsymbol{u}\,.\boldsymbol{g}=-L$$

Simplifying using $p\boldsymbol{\nabla}.\boldsymbol{u}+\boldsymbol{u}\,.\boldsymbol{\nabla}p=\boldsymbol{\nabla}.(p\boldsymbol{u})$ and $\boldsymbol{g}=-\boldsymbol{\nabla}\psi$

$$\rho\frac{de}{dt}+\rho\,\boldsymbol{u}\,.\frac{d\boldsymbol{u}}{dt}+\boldsymbol{\nabla}.(p\boldsymbol{u})+\rho\boldsymbol{u}\,.\boldsymbol{\nabla}\psi=-L$$

And $\rho\frac{de}{dt}+\rho\,\boldsymbol{u}\,.\frac{d\boldsymbol{u}}{dt}=\rho\frac{de}{dt}+\rho\frac{d}{dt}\left(\frac{u^2}{2}\right)=\rho\frac{d}{dt}\left(e+\frac{u^2}{2}\right)$, leaving:

$$\rho\frac{d}{dt}\underbrace{\left(e+\frac{u^2}{2}\right)}_{\text{Note that this is an energy per unit mass (specific energy)}}+\boldsymbol{\nabla}.(p\boldsymbol{u})+\rho\boldsymbol{u}\,.\boldsymbol{\nabla}\psi=-L$$

But we can simplify this even further…

Noting that the time derivative is a full one, and using $A=e+\frac{u^2}{2}$:

$$\rho\frac{dA}{dt}+\boldsymbol{\nabla}.(p\boldsymbol{u})+\rho\boldsymbol{u}\,.\boldsymbol{\nabla}\psi=-L$$

Where $\rho\frac{dA}{dt}=\frac{d(\rho A)}{dt}-A\,\frac{d\rho}{dt}$ and $\frac{d\rho}{dt}=-\rho\boldsymbol{\nabla}.\boldsymbol{u}$ so

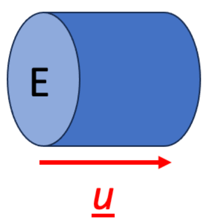
$$\rho\frac{dA}{dt}=\frac{d(\rho A)}{dt}-A\,(-\rho\boldsymbol{\nabla}.\boldsymbol{u})=\frac{d(\rho A)}{dt}+(A\rho)\boldsymbol{\nabla}.\boldsymbol{u}=\left(\frac{\partial(\rho A)}{\partial t}+\boldsymbol{u}.\boldsymbol{\nabla}(\rho A)\right)+(A\rho)\boldsymbol{\nabla}.\boldsymbol{u}$$

$$\text{leaving }\rho\frac{dA}{dt}=\frac{\partial(\rho A)}{\partial t}+\boldsymbol{\nabla}.\big(\boldsymbol{u}(A\rho)\big)$$

so, we then find:

$$\frac{\partial(\rho A)}{\partial t}+\boldsymbol{\nabla}.\big(\boldsymbol{u}(A\rho)\big)+\boldsymbol{\nabla}.(p\boldsymbol{u})+\rho\boldsymbol{u}\,.\boldsymbol{\nabla}\psi=-L$$

$$\underbrace{\frac{\partial}{\partial t}\left(\rho e+\rho\frac{u^2}{2}\right)}+\boldsymbol{\nabla}.\left(\boldsymbol{u}\left(\rho e+\rho\frac{u^2}{2}\right)\right)+\boldsymbol{\nabla}.(p\boldsymbol{u})+\rho\boldsymbol{u}\,.\boldsymbol{\nabla}\psi=-L$$



Unit check: $kg\,m^{-3}m^2s^{-2}=kgm^{-1}s^{-2}$ ie energy/volume

$\left(\rho\frac{u^2}{2}\right)\boldsymbol{u}$: energy × velocity/volume = energy × time/area ie an energy flux

Trying to simplify even further…

$$\nabla.(\rho\psi\boldsymbol{u}) = (\rho\psi)\,\nabla.\boldsymbol{u} + \boldsymbol{u}.\nabla(\rho\psi)$$

$$\nabla.(\rho\psi\boldsymbol{u}) = (\rho\psi)\,\nabla.\boldsymbol{u} + \psi\,\boldsymbol{u}.\nabla\rho + \rho\,\boldsymbol{u}.\nabla\psi$$

$$\nabla.(\rho\psi\boldsymbol{u}) = \psi(\rho\,\nabla.\boldsymbol{u} + \boldsymbol{u}.\nabla\rho) + \rho\,\boldsymbol{u}.\nabla\psi$$

$$\nabla.(\rho\psi\boldsymbol{u}) = \psi(\nabla.(\rho\boldsymbol{u})) + \rho\,\boldsymbol{u}.\nabla\psi$$

$$\nabla.(\rho\psi\boldsymbol{u}) = \psi\left(-\frac{\partial\rho}{\partial t}\right) + \rho\,\boldsymbol{u}.\nabla\psi$$

And rearranging:

$$\rho\,\boldsymbol{u}.\nabla\psi = \psi\frac{\partial\rho}{\partial t} + \nabla.(\rho\psi\boldsymbol{u})$$

Which we can put into the energy equation:

$$\frac{\partial}{\partial t}\left(\rho e + \rho\frac{u^2}{2}\right) + \nabla.\left(\boldsymbol{u}\left(\rho e + \rho\frac{u^2}{2}\right)\right) + \nabla.(p\boldsymbol{u}) + \psi\frac{\partial\rho}{\partial t} + \nabla.(\rho\psi\boldsymbol{u}) = -L$$

$$\frac{\partial}{\partial t}\left(\rho e + \rho\frac{u^2}{2}\right) + \psi\frac{\partial\rho}{\partial t} + \nabla.\left(\boldsymbol{u}\left(\rho e + \rho\frac{u^2}{2}\right)\right) + \nabla.(p\boldsymbol{u}) + \nabla.(\rho\psi\boldsymbol{u}) = -L$$

$$\frac{\partial}{\partial t}\left(\rho e + \rho\frac{u^2}{2}\right) + \psi\frac{\partial\rho}{\partial t} + \nabla.\left(\boldsymbol{u}\left(\rho e + \rho\frac{u^2}{2} + p + \rho\psi\right)\right) = -L$$

<u>Reminder:</u> $\nabla.\boldsymbol{u}$ is a velocity emerging from a unit volume. $\nabla.(a\boldsymbol{u})$ is a rate of flow (amount/second) of "a" out of the unit volume.

So, in steady state, the energy equation is:

$$\boxed{\nabla.\left(\boldsymbol{u}\left(\rho e + \rho\frac{u^2}{2} + p + \rho\psi\right)\right) = -L}$$

Where $\rho e + p = \frac{\gamma}{\gamma-1}p$, and is the entropy.

In this course we won't deal with time dependence.

Within some volume, the net effect (L) of the sources and sinks of energy is equal (in a steady state) to the energy through the surface i.e.

$$\iiint_V -L\,dV = \oiint_S \boldsymbol{u}\,.\left(\rho\frac{u^2}{2} + \rho e + p + \rho\psi\right)d\boldsymbol{S}$$

<u>Aside on equation of state</u>:
In general, $p = p(\rho, T)$ and for an ideal gas: $p = nk_BT = mk_BT/\rho$ where $\rho = mn.$ For a fully ionised hydrogen plasma, $n = n_e + n_p = 2n_e$ and $\rho = m_pn_p + m_en_e \approx n_em_p$

n: number of particles per unit volume and m: mean particle mass

Barotropic Equations of state $p(\rho)$:
This means that p can only be written as a function of ρ, (e.g. for an ideal gas, ρ must be related to T)

Two options:

1. **Isothermal**: $T = constant$ so $p \propto \rho$
   For this to be a good approximation we require:
   - Temperature for thermal equilibrium isn't very sensitive to heating/cooling.
   - In time-dependent cases, there is time for the system to reach this constant T, thermal equilibrium.
2. **Adiabatic**: $p = k\rho^{\gamma}$
   This is derived from the ideal gas laws on assumption of no heat exchange $(dQ = 0)$ i.e. adiabatic.
   - A fluid element behaves adiabatically if k is constant as the fluid elements properties change.
   - *Isentropic fluid*: one in which all elements have the same k.

Summary of the 3 fluid equations:

1. Mass conservation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}.(\rho\boldsymbol{u}) = 0$$

2. Momentum conservation

$$\rho\frac{d\boldsymbol{u}}{dt} = -\nabla p + \rho\boldsymbol{g} + \boldsymbol{F}_{ext}$$

3. Energy conservation

$$\boldsymbol{\nabla}.\left(\left(\rho e + \rho\frac{u^2}{2} + p + \rho\psi\right)\boldsymbol{u}\right) = -L$$

For a closed system, $L = 0$. And in steady state, i.e. no E flowing in or out: E is conserved.

Equations of state can be written to describe systems, but we can only talk about them for collisional fluids (then pressure means something) and we use an equation which relates p to other thermodynamical properties.

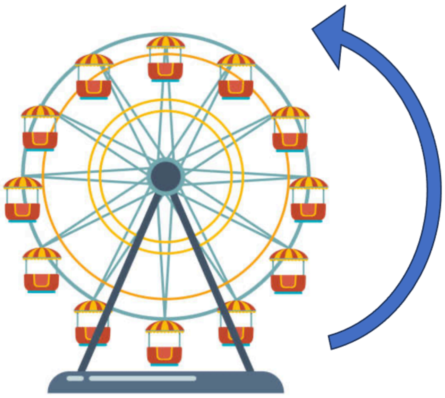
Lecture 7

Vorticity and viscosity:
Good website: earth.nullschool.net – see the weather real time.

What is vorticity? ("angular momentum and all that")

Vorticity is the tendency of a fluid particle to rotate about an axis through its <u>own</u> centre of mass.
NOTE: each fluid particle must rotate.

Vorticity is just an expression on angular momentum in a fluid.

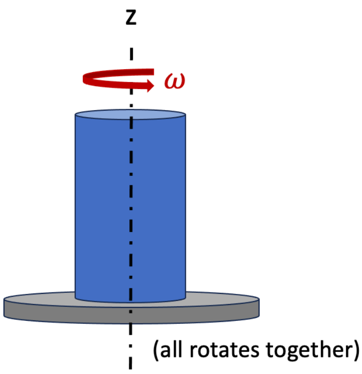


Mathematically: $$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$$

A rough analogy: the Ferris wheel has rotational motion, but the passengers don't.
⇒ the wheel has vorticity, but the people aren't rotating about their own centre of mass, therefore *they* don't.

A parcel of fluid must be rotating about its own centre of mass to have vorticity.

<u>Example 1:</u> solid body rotation



(e.g. a glass of water placed at the centre of a turntable)

$$\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}$$

Our body is cylindrical and thus cylindrical polars will be most appropriate:
$u_p = \Omega r,\ u_r = u_z = 0$ (only rotating "angularly")

This is a simple system; every parcel has the same vorticity.
Calculating:

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} - \boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \overset{=0}{\cancel{(\boldsymbol{r}.\boldsymbol{\nabla})\boldsymbol{\Omega}}} + (\boldsymbol{\nabla}.\boldsymbol{r})\boldsymbol{\Omega} - \overset{=0}{\cancel{(\boldsymbol{\nabla}.\boldsymbol{\Omega})\boldsymbol{r}}} - \overset{=0}{\cancel{(\boldsymbol{\Omega}.\boldsymbol{\nabla})\boldsymbol{r}}}$$

$(\boldsymbol{\nabla}.\boldsymbol{\Omega}) = 0$ because $\boldsymbol{\Omega} = constant$

$\boldsymbol{\Omega} = \Omega\,\hat{\boldsymbol{z}}$ (angular velocity is along the direction of rotation).

Therefore, $\boldsymbol{\omega} = -\left(\boldsymbol{\Omega}\frac{\partial}{\partial z}(x,y)\right) + \left(\boldsymbol{\Omega}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right)\right)$

$\boldsymbol{\omega} = 2\boldsymbol{\Omega}$ for this system (solid body)

c.f. specific angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = (\boldsymbol{r}.\boldsymbol{r})\boldsymbol{\Omega} - (\boldsymbol{r}.\boldsymbol{\Omega})\boldsymbol{r} = r^2\boldsymbol{\Omega}$$

So $$\boldsymbol{L} = \frac{r^2\boldsymbol{\Omega}}{\boldsymbol{2}}$$

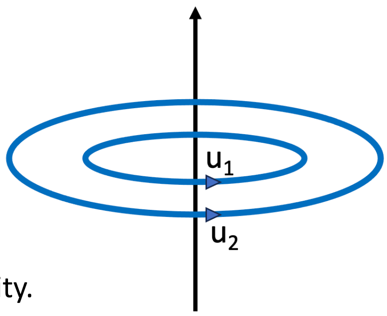
$\boldsymbol{L}$ lies along the same direction as the axis of rotation and the angular velocity.

!! very important !! for solid body rotation we find that vorticity is just twice the angular velocity.

$$\boxed{\boldsymbol{\omega} = 2\boldsymbol{\Omega}}$$

This is very similar to L which is related to vorticity.

Solid body is the simplest type because in this case the fluid element doesn't distort as it moves.



Example 2: water down a plughole

There is only a u_ϕ component:

$$\boldsymbol{u} = (u_r, u_\phi, u_z) = \left(0, \frac{k}{r}, 0\right)$$

Every fluid parcel moves in a circle, but with a different azimuthal velocity.

In cylindrical coordinates:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \phi} - \frac{\partial u_\phi}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\phi) - \frac{1}{r}\frac{\partial u_r}{\partial \phi}\right)\hat{\boldsymbol{z}}$$

Here:

$$\boldsymbol{\nabla} \times \boldsymbol{u} = \left(-\frac{\partial u_\phi}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\phi)\right)\hat{\boldsymbol{z}}$$

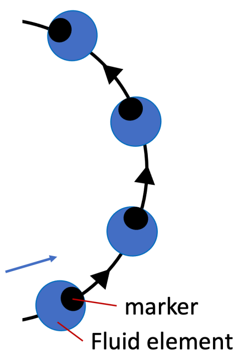
$$\boldsymbol{\nabla} \times \boldsymbol{u} = \left(\frac{1}{r}\frac{\partial}{\partial r}(k)\right)\hat{\boldsymbol{z}}$$

$$\omega_r = \omega_\phi = 0 \text{ and } \omega_z = 0 \ \ \forall \boldsymbol{r} \ except \ \boldsymbol{r} = \boldsymbol{0}$$

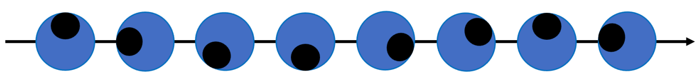
We have a singularity at $\boldsymbol{r} = \boldsymbol{0}$. This is because we've ignored viscosity, leading to an undefined ω. (singularity in the velocity field)

Finding singularities usually means we've missed some important physics!

For solid body rotation:

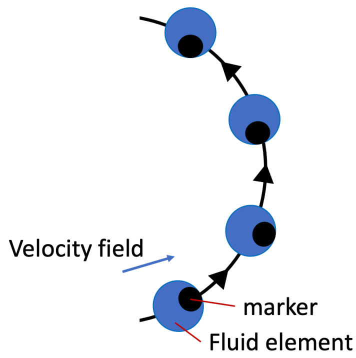


For true solid body rotation, the marker is always aligned to the path traced out as it was initially.



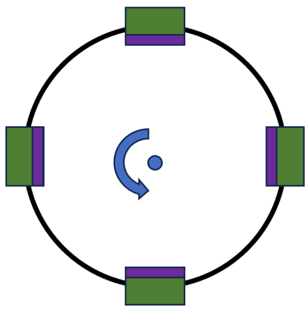
i.e. it rotates about its centre of mass.

But this is not what happens in example 2. What we see there is in fact solid body + shear.



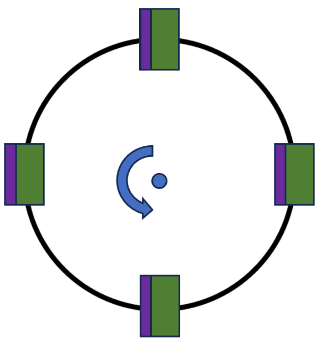
Recall: $u_\phi = \frac{k}{r}$ so increasing r means decreasing u_ϕ. The rotations add up to give a flow where the time to rotate is identical as to trace the circle.
$\Rightarrow$ no vorticity anywhere **but** the axis.

So circular motion doesn't necessarily imply vorticity!



Rigid body rotation: each parcel of fluid changes its orientation as it moves.

$\omega \neq 0$ often.



Circulation without rotation: each parcel maintains the same orientation, even though it moves in a circle.

$\omega = 0$ often.

NOTE: we need to know the form of $\boldsymbol{u}$ to know if vorticity exists.

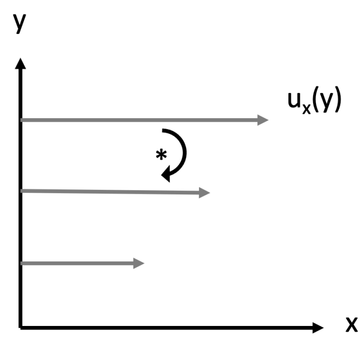
"Vortex sheets": vorticity with no circular motion (macroscopically)

Example 3: vortex sheets

Shear flow: $\boldsymbol{u} = u_x(y)\hat{\boldsymbol{x}}$ and $u_y = u_z = 0$.

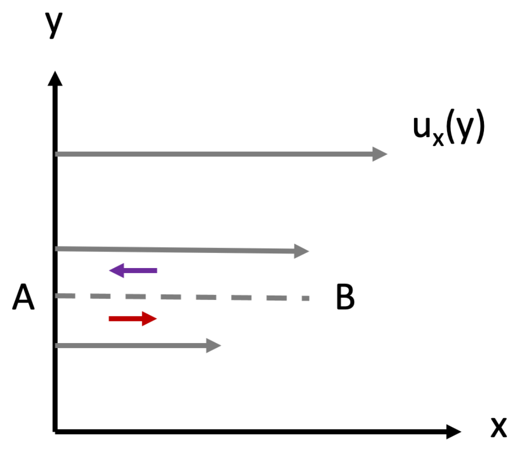
Vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} = \left(\underbrace{\cancel{\frac{\partial u_z}{\partial y}}}_{=0} - \underbrace{\cancel{\frac{\partial u_y}{\partial z}}}_{=0}\right)\hat{\boldsymbol{x}} + \left(\underbrace{\cancel{\frac{\partial u_x}{\partial z}}}_{=0} - \underbrace{\cancel{\frac{\partial u_z}{\partial x}}}_{=0}\right)\hat{\boldsymbol{y}} + \left(\underbrace{\cancel{\frac{\partial u_y}{\partial x}}}_{=0} - \frac{\partial u_x}{\partial y}\right)\hat{\boldsymbol{z}}$

$$\boldsymbol{\omega} = \left(-\frac{\partial u_x(y)}{\partial y}\right)\hat{\boldsymbol{z}}$$



i.e. $\boldsymbol{\omega} = \left(0{,}0, -\frac{\partial u_x}{\partial y}\right)$
the larger $u_x(y)$ is, the larger ω is.
→ The large-scale flow is not at all circular. Macroscopically it's linear but there is vorticity.
→ A paddle wheel in this flow would rotate.

<u>What is viscosity?</u>
Viscosity allows us to get a solution for example 2 that is continuous and well behaved. Currently, our solution to example 2 is for an inviscid fluid so is missing physics. The viscosity clears up the discontinuity.

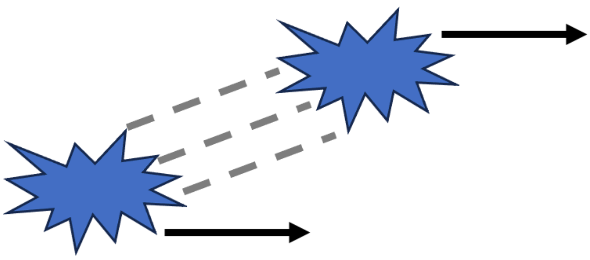


→ the faster fluid above plane AB drags the fluid below forwards (red arrow on the graph).
→ the slower fluid below AB drags the fluid above backwards (purple).
→ the forces acting on AB are shown by the arrows (the forces are equal but opposite). They're drawn on the side of the surface to which they act.

Viscosity is responsible for this internal stress (F/A).

These forces are being applied only because of viscosity. If we had an inviscid fluid, the elements above and below would be completely unaware of those elements around it.

Viscosity tells us how "sticky" the fluid particles are:



For a Newtonian fluid (viscosity is independent of the velocity, though may vary with p, T), the force per unit area (stress) is:

Coefficient of shear viscosity [kg/m/s]

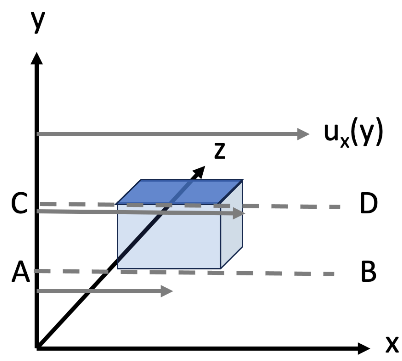
$$\boxed{\tau = \mu \frac{\partial u}{\partial y}}$$

$$\tau = F/A$$

Hence, for a given flow $\boldsymbol{u}$, the higher the viscosity, the greater the stress.

Lecture 8

Now take a whole fluid element (not just a plane). The net viscous force is the difference of the viscous forces on both sides.



For a Newtonian fluid, with viscosity not being a function of velocity:

$$\tau = \mu \frac{\partial u}{\partial y}$$

So below $\tau = \mu \frac{\partial u}{\partial y}\Big|_y$ and above $\tau = \mu \frac{\partial u}{\partial y}\Big|_{y+\delta y}$

The viscous force per element of volume then is:

$$F = \tau\, A$$
$$\Delta F = \Delta\tau\; \delta x \delta z$$
$$\Delta F = \left(\mu \frac{\partial u}{\partial y}\bigg|_{y+\delta y} - \mu \frac{\partial u}{\partial y}\bigg|_{y} \right)\; \delta x \delta z$$

Using the definition of differentiation:

$$\frac{dQ}{dx} \equiv \lim_{\delta x \to 0} \left\{ \frac{Q(x+\delta x) - Q(x)}{\delta x} \right\}$$

Hence $\left(\mu \frac{\partial u}{\partial y}\Big|_{y+\delta y} - \mu \frac{\partial u}{\partial y}\Big|_{y} \right) = \mu \left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\; \delta y \right)$

Here, we've treated μ as a constant. It technically isn't however it often acts as one, especially locally.

$$\Delta F = \mu \left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\; \delta y \right)\; \delta x \delta z$$
$$\Delta F = \mu \left(\frac{\partial^2 u}{\partial y^2} \right) V$$

So, for the y direction

$$\frac{\Delta F}{V} = F_V = \mu \left(\frac{\partial^2 u}{\partial y^2} \right).$$

By analogy we can get similar results for the x and z directions.

$$F_V = \mu \left(\frac{\partial^2 u}{\partial x^2} \right)$$

$$F_V = \mu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

Combining these results:

$$F_V = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\boldsymbol{F_V} = \rho\nu\,\nabla^2\boldsymbol{u}$$

For an incompressible fluid, with $\nu = \mu/\rho$ and is called the "kinematic viscosity" [m^2/s].

<u>Equation of motion</u>:

Using this equation, we can alter the EOM to include viscosity:

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{g} + \nu\nabla^2\boldsymbol{u}$$

Numerically, it is very difficult to solve/evaluate $\nu\nabla^2\boldsymbol{u}$.

Every time we differentiate, we lose information, plus ν is often very small. Due to these things, it can be difficult to evaluate efficiently or accurately.

So sometimes we don't want to include this term. When should we put this term into the EOM, and when is it safe to ignore it?

<u>Reynold's number</u>:

Take the steady state EOM with $\boldsymbol{g} = \boldsymbol{0}$:

$$\underbrace{(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u}}_{\text{Inertial term}} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \underbrace{\nu\nabla^2\boldsymbol{u}}_{\text{viscous term}}$$

From this, we can define the Reynold's number as the ratio of the inertial term to the viscous terms.

$$R = \frac{(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u}}{\nu\nabla^2\boldsymbol{u}}$$

And we calculate it using dimensional analysis:

$$R = \frac{(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u}}{\nu\nabla^2\boldsymbol{u}} \qquad \begin{array}{l} \sim u^2/L \\ \sim \nu u/L^2 \end{array}$$

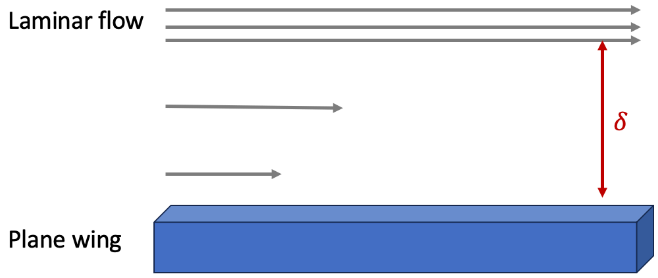
$$R = \frac{uL}{\nu}$$

There are two extreme cases here:

- $R \gg 1$. In this case the inertial term dominates, and viscosity of the fluid is unimportant. Therefore, we may ignore it. Looking at the equation we see that this occurs for large u or L (length-scales)
- $R \sim 1$. In this case the viscosity is important and is on the same order as the inertial term. Hence, we cannot ignore it.

So, length-scales are very important, a small insect flying around the room with air conditioning experiences a more viscous fluid than a human.

Hence, bees fly through a low Reynold's number (viscous fluid) whilst aeroplanes fly through a flow with a high Reynold's number (less viscous). However, aeroplanes do experience some viscosity at the boundary layer.



Over a length-scale δ, the flow slows down to zero at the boundary layer. Hence, there must be some viscosity.

If the aircraft is well designed, we should have a thin smooth boundary layer – as this prevents the breaking up of the boundary layer. When the boundary layer breaks up, it shed vortexes which makes the movement inefficient.

We want to know how this boundary layer behaves.

<u>Boundary layer thickness</u>:

- Intuitively we expect the thickness to depend on viscosity of the fluid.

$$(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u} \approx -\frac{1}{\rho}\boldsymbol{\nabla}p$$

Which has dimensions:

$$|(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u}| \sim \frac{u^2}{L}$$

This is for within the body of the flow. Within the boundary layer, however, the viscous term becomes important and so it has dimensions:

$$(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u} \approx \nu\nabla^2\boldsymbol{u}$$
$$|(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u}| \sim \frac{u^2}{L} \sim \frac{\nu u}{L^2} = \frac{\nu u}{\delta^2}$$

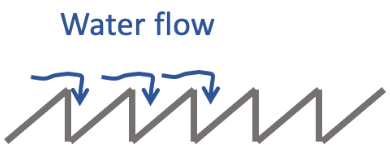
For the viscous term to compete with the "large-scale" inertial term, these two must be equal:

$$\frac{u^2}{L} \sim \frac{\nu u}{\delta^2}$$
$$\frac{\delta}{L} \sim \left(\frac{uL}{\nu}\right)^{-\frac{1}{2}} = \frac{1}{\sqrt{R}}$$

Which tells us that

Low viscosity → high Reynold's number → thin boundary layer

Shark skin:



This prevents the boundary layer from breaking up. Water is pulled back down before it can escape, forcing the boundary layer to reattach. This makes their motion more efficient as no vortexes are shed.

Note: viscosity is important because without it we can't fully understand vorticity.

Bernoulli's equation:
This is really just an expression of energy conservation!

We will leave out viscosity and find that we cannot explain things…
The EOM without viscosity is

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{g} - \boldsymbol{\nabla}\psi.$$

Now, for a barotropic equation of state $p(\rho)$:

$$\begin{aligned}\frac{1}{\rho}\boldsymbol{\nabla}p &= \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial p}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial p}{\partial z}\hat{\boldsymbol{z}}\right)\\ &= \frac{1}{\rho}\frac{\partial p}{\partial \rho}\left(\frac{\partial \rho}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial \rho}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial \rho}{\partial z}\hat{\boldsymbol{z}}\right)\\ &= \frac{1}{\rho}\frac{\partial p}{\partial \rho}\boldsymbol{\nabla}\rho\\ &= \frac{\partial}{\partial \rho}\left[\int \frac{\partial p}{\partial \rho}\frac{d\rho}{\rho}\right]\boldsymbol{\nabla}\rho\\ &= \frac{\partial}{\partial \rho}\left[\int \frac{dp}{\rho}\right]\boldsymbol{\nabla}\rho\\ &= \boldsymbol{\nabla}\left[\int \frac{dp}{\rho}\right]\end{aligned}$$

So…

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}\left[\int \frac{dp}{\rho} + \psi\right].$$

Using the vector identity:

$$(\boldsymbol{u}\,.\,\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\left(\frac{u^2}{2}\right) - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})$$

We find:

$$\begin{aligned}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{u^2}{2}\right) - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) &= -\boldsymbol{\nabla}\left[\int \frac{dp}{\rho} + \psi\right]\\ \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u}\times\boldsymbol{\omega} &= -\boldsymbol{\nabla}\left[\int \frac{dp}{\rho} + \psi\right] - \boldsymbol{\nabla}\left(\frac{u^2}{2}\right)\end{aligned}$$

Specific energy terms.

$$\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega} = -\nabla \overbrace{\left[\frac{u^2}{2} + \int \frac{dp}{\rho} + \psi\right]}$$

This is the general form of the Bernoulli equation without viscosity. This form isn't very useful though…
This equation is still just a rate of change of momentum = sum of forces. We've only done some algebra; the physics hasn't changed!

Two special cases of the equation:
1. take a steady state and dot product with u:

$$\boldsymbol{u}.\nabla\left[\frac{u^2}{2} + \int \frac{dp}{\rho} + \psi\right] = 0$$

$$\boldsymbol{u}.\nabla H = 0$$

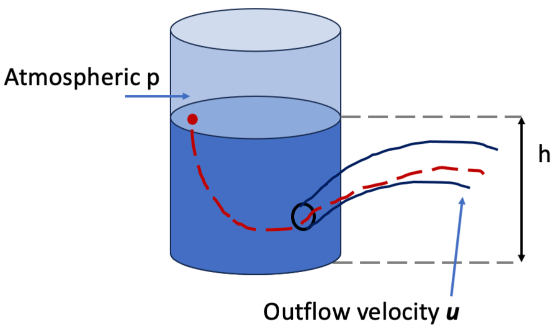
Where $H \equiv \frac{u^2}{2} + \int \frac{dp}{\rho} + \psi$ and is "Bernoulli's constant" *which is constant along streamlines*.

If we move in that direction at that velocity, ∇H is the change in H that I see. Here that is zero! It is conserved in my direction of motion. It may differ between streamlines, but once you're on one it doesn't change – an expression of conservation of energy.

- The mechanical energy for a fluid element along a specific streamline is the same along its path.
- This form applies for a steady state (path and streamline are the same here).

2. if the flow is steady and the vorticity is zero (irrotational flow) then $\nabla H = 0$.
- *H* is constant **everywhere** i.e. the same on every streamline (every streamline has the same energy).
- All the parcels of fluid not only conserve their own total mechanical energy *H*, but they also have the same *H* as all their neighbours. This is an equalising flow.

Example 1: for an incompressible fluid, where density is uniform:



Since density is constant:

$$\int \frac{dp}{\rho} = \frac{1}{\rho}\int dp = \frac{p}{\rho}$$

$$H = \frac{u^2}{2} + \int \frac{dp}{\rho} + \psi = \frac{u^2}{2} + \frac{p}{\rho} + \psi$$

For a parcel of fluid following the dotted line:
H must be conserved

$$0 + \frac{p_{top}}{\rho} + \psi_{top} = \frac{u^2}{2} + \frac{p_{bottom}}{\rho} + \psi_{bottom}$$

Pressure hasn't changed so:

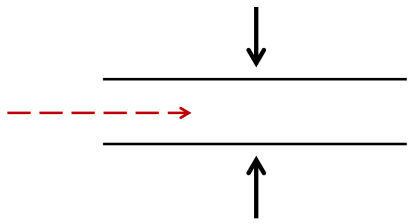
$$\psi_{top} = \frac{u^2}{2} + \psi_{bottom}$$

$$\frac{u^2}{2} = \psi_{top} - \psi_{bottom} = gh$$

$$u = \sqrt{2gh}$$

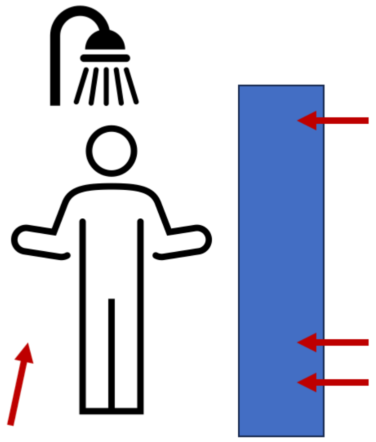
But we have a problem here. Why doesn't the fluid reach the same height as it fell from?
Answer: we haven't considered air or water friction. We've missed out viscosity!

Example 2: if we blow between two pieces of paper, where the velocity is large, the pressure is small (as you're blowing the air away). This pressure difference between the top and bottom of the paper causes the two pieces to be pulled together.



$$H = \frac{u^2}{2} + \frac{p}{\rho}$$

Example 3: why does the shower curtain cling to you?



The warm air rises, causing changes in the pressure and the curtain is pushed against you.

If we take the curl of Bernoulli's equation:

$$\boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega}\right) = \boldsymbol{\nabla} \times \left(-\boldsymbol{\nabla}\left[\frac{u^2}{2} + \int \frac{dp}{\rho} + \psi\right]\right) = \boldsymbol{0}$$

$$\frac{\partial(\boldsymbol{\nabla} \times \boldsymbol{u})}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{0}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{0}$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega})$$ Helmholtz equation.

This equation tells us that if initially the vorticity Is zero, then it must always be zero i.e. we cannot create vorticity.
We know that this is factually incorrect – we can produce vorticity from nothing. This mistake is because we have not accounted for viscosity.

If we repeat the above steps but this time include viscosity:

$$\boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \times \boldsymbol{\omega}\right) = \boldsymbol{\nabla} \times \left(-\boldsymbol{\nabla}\left[\frac{u^2}{2} + \int \frac{dp}{\rho} + \psi - \nu \boldsymbol{\nabla}.\boldsymbol{u}\right]\right)$$

$$\frac{\partial(\boldsymbol{\nabla} \times \boldsymbol{u})}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{\nabla} \times \left(-\boldsymbol{\nabla}\left[\frac{u^2}{2} + \int \frac{dp}{\rho} + \psi\right]\right) + \boldsymbol{\nabla} \times \left(-\boldsymbol{\nabla}(-\nu \boldsymbol{\nabla}.\boldsymbol{u})\right)$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \boldsymbol{\nabla} \times (\nu \nabla^2 \boldsymbol{u})$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \boldsymbol{\nabla} \times (\nabla^2 \boldsymbol{u})$$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \nabla^2 (\boldsymbol{\nabla} \times \boldsymbol{u})$$

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}}$$

Eulerian: if I stand still, the vorticity changes in time due to these two terms.

What the flow does to vorticity.

$\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega})$: advective term - carries vortex lines around.

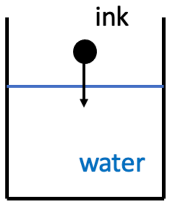
- describes how vortex lines are pushes around by the flow, they can be stretched and twisted (by the velocity).

What viscosity does to vorticity.

$\nu \nabla^2 \boldsymbol{\omega}$: diffusion term – explains how vorticity is created.

- This term describes how vorticity is dissipated by viscosity.

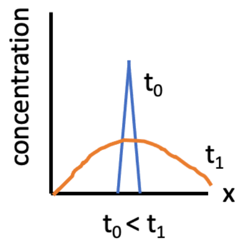
Aside on diffusion equations:
How does the concentration of a drop of ink change with time due to $\partial^2/\partial x^2$ when added to a cup of water?
Big changes in spatial gradients cause fast evolution in time.



$$\frac{\partial \boldsymbol{\omega}}{\partial t} \approx \nu \nabla^2 \boldsymbol{\omega}$$

Dimensions:

$$\frac{\omega}{t} \approx \frac{\nu \omega}{L^2}$$
$$t \approx \frac{L^2}{\nu}$$



<u>Question 6:</u> Write down H for (a) adiabatic and (b) isothermal equations of state.

<u>Answer 6</u>:

(a) $dQ = 0, p = k\rho^\gamma$

$$H = \frac{u^2}{2} + \psi + \int \frac{dp}{\rho}$$
$$\frac{\partial p}{\partial \rho} = k\,\gamma\,\rho^{\gamma-1} = k\,\gamma\,\frac{\rho^\gamma}{\rho} = \frac{p}{\rho}\gamma$$

Therefore

$$\int \frac{dp}{\rho} = \int p\frac{\gamma}{\rho^2}d\rho = \gamma\,k\int \rho^{\gamma-2}d\rho = \gamma\frac{k}{\gamma-1}\rho^{\gamma-1} = \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}$$
$$H = \frac{u^2}{2} + \psi + \left(\frac{\gamma}{\gamma-1}\right)\frac{p}{\rho}$$

(b) $T = constant, p = \frac{k\rho T}{m}$

$$\frac{dp}{d\rho} = \frac{kT}{m}$$

$$\int dp/\rho = \frac{kT}{m}\int d\rho/\rho = \frac{kT}{m} ln\rho = c_s^2 ln\rho$$

Hence,

$$H = \frac{u^2}{2} + \psi + c_s^2 ln\rho$$